Advances in Mathematics 292 (2016) 707–754

Contents lists available at ScienceDirect

Advances in Mathematics

www.elsevier.com/locate/aim

Symmetric powers in abstract homotopy categories

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A R T I C L E I N F O

Article history:
Received 21 February 2015
Received in revised form 16 December 2015
Accepted 26 January 2016
Communicated by the Managing Editors of AIM

MSC:
14F42
18D10
18G55

Keywords:
Model structures
Symmetric monoidal model categories
Symmetric powers
Generating cofibrations
Localization of model categories
Symmetric spectra
$\mathbb{A}^1$-homotopy theory
Motivic spaces and motivic symmetric spectra

A B S T R A C T

We study symmetric powers in the homotopy categories of abstract closed symmetric monoidal model categories, in both unstable and stable settings. As an outcome, we prove that symmetric powers preserve the Nisnevich and étale homotopy type in the unstable and stable motivic homotopy theories of schemes over a base. More precisely, if f is a weak equivalence of motivic spaces, or a weak equivalence between positively cofibrant motivic spectra, with respect to the Nisnevich or étale topology, then all symmetric powers $\mathrm{Sym}^n(f)$ are weak equivalences too. This gives left derived symmetric powers in the corresponding motivic homotopy categories of schemes over a base, which aggregate into a categorical λ-structures on these categories.

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http://dx.doi.org/10.1016/j.aim.2016.01.011

Contents

1. Introduction

In topology, symmetric powers preserve homotopy type of CW-complexes, which is at the heart of the fundamental Dold–Thom theorem connecting the homology of a complex to the homotopy groups of its infinite symmetric power, see [2]. A natural question is to which extent such phenomena could be true in the motivic $\mathbb{A}^1$-homotopy theory of schemes over a base? The first steps in this direction were made in the pioneering work [21]. In [23] Voevodsky developed a motivic theory of symmetric powers, good enough to construct motivic Eilenberg–MacLane spaces needed for the proof of the Bloch–Kato conjecture. His symmetric powers depend on symmetric powers of schemes presenting motivic spaces. The aim of this paper is to develop a purely homotopical theory of symmetric powers in an abstract symmetric monoidal model category, and to give an affirmative answer to the question when symmetric powers preserve weak equivalences in such a category, working out the unstable and stable settings separately.

More technically, working in a closed symmetric monoidal model category $\mathscr{C}$, we address the following two fundamental questions in the paper. Whether left derived symmetric powers exist in the homotopy category $Ho(\mathscr{C})$ and, if they do, whether they aggregate into a (categorical) λ-structure on the homotopy category of $\mathscr{C}$? The latter concept means that, given a morphism in $Ho(\mathscr{C})$, there exists a tower connecting the derived symmetric powers of the domain and codomain, whose cones can be computed by the Künneth rule. A categorical λ-structure is then a system of Künneth towers, functorial on morphisms in $Ho(\mathscr{C})$. If the categorical λ-structure preserves compact objects in $\mathscr{C}$, then it induces a usual λ-structure on the K_0-ring of the Waldhausen category of cofibrant compact objects in $\mathscr{C}$.

We develop a general machinery to deal with that kind of questions in $\mathscr{C}$, and in symmetric spectra over $\mathscr{C}$. The methods for the stable and unstable cases are surprisingly different. In the unstable setting, we introduce the notion of symmetrizable cofibrations and study how symmetrizability behaves under cofibrant generation and localization in the sense of [8]. With this aim, we provide quite a general condition on a left derived functor so that it factors through the localized homotopy category. The main type of localization is the contraction of a diagonalizable interval in $\mathscr{C}$. In the stable setting we construct a positive model structure on the category of symmetric spectra, in which weak equivalences are the usual stable weak equivalences and all cofibrations are isomorphisms on level zero. Our positive model structure is an utmost generalization of the topological positive model structure constructed in [3], and the motivic positive model structure introduced in [9]. Positive model structures are the main tool in the study of symmetric powers of abstract symmetric spectra over $\mathscr{C}$.

Our main destination is, however, the motivic $\mathbb{A}^1$-homotopy theory of schemes, and we anticipate numerous applications of our methods and results in arithmetic and geometry through that theory. For the present, we prove the following two theorems giving positive answers to the questions above in the unstable and stable motivic homotopy theory of schemes over a base:

Theorem A. *Symmetric powers preserve the Nisnevich and étale homotopy type of motivic spaces, left derived symmetric powers exist in the unstable motivic homotopy category of schemes over a base and aggregate into a categorical λ-structure on it.*

Theorem B. *Symmetric powers preserve stable weak equivalences between positively cofibrant motivic symmetric spectra, left derived symmetric powers exist in the motivic stable homotopy category of schemes over a base and aggregate into a categorical λ-structure on it. The left derived symmetric powers of motivic spectra coincide with the corresponding homotopy symmetric powers.*

In a broader context, homotopical theory of symmetric powers has many potential applications. For example, it can be used to construct a model structure on commutative monoids, and a global model structure for ultra-commutative monoids in a symmetric monoidal model category, see [24] and [20]. The results and technique of the present paper are further developed and extended to the setting of abstract symmetric operads in [18]. In [17] the obtained results are used to prove that the motivic rational homotopy type of symmetric powers of motivic spectra and motivic spectra of geometric symmetric powers coincide. Finally, in [7] we used our theory to discover a new phenomenon in Chow groups of algebraic varieties over a field.

Now we give a road map of the paper. We start by introducing the notion of symmetrizable (trivial) cofibrations in $\mathscr{C}$. To study left derived symmetric powers, it would be natural to consider (trivial) cofibrations whose symmetric powers are again (trivial) cofibration. However, we need to introduce a stronger property so that it becomes invariant under compositions and pushouts. Loosely speaking, (trivial) cofibrations are symmetrizable (Definition 3) if they are stable under taking colimits of the action of symmetric groups on their pushout products in $\mathscr{C}$. If cofibrations are symmetrizable, then one can associate, to a cofibre triangle in $\mathscr{C}$, a tower of cofibrations connecting symmetric powers of the vertices of the triangle, and whose cones can be computed by Künneth's rule. Such Künneth towers can be viewed as a sort of categorification of λ-structures in commutative rings (Definition 24), and give a powerful tool to work out symmetric powers (Theorem 22). If trivial cofibrations between cofibrant objects are symmetrizable, then symmetric powers preserve weak equivalences between cofibrant objects and so admit their left derived endofunctors on $\mathscr{C}$ (Theorem 25). When $\mathscr{C}$ is cofibrantly generated by the set of generating cofibrations I and the set of trivial generating cofibrations J, and if the sets I and J are both symmetrizable, then all cofibrations and trivial cofibrations in $\mathscr{C}$ are symmetrizable (Theorem 7 and Corollary 9). This is useful in applications to concrete cofibrantly generated monoidal model categories, and will be applied to symmetric spectra in Section 9. If, in addition, symmetric powers of cofibrant replacements of morphisms in a set of morphisms S are S-local equivalences, then trivial cofibrations between cofibrant objects in the left localization $\mathscr{C}_S$ are symmetrizable (Theorem 33). To show this, we give a condition on a left derived functor (which might not have right adjoint) to factor it through the localized homotopy category (Theorem 29). This result can be applied to a broad range of Bousfield localizations. An important particular case is when S-localization is a contraction of a diagonalizable interval into a point (Theorem 42).

In topology, i.e. when $\mathscr{C}$ is the category of simplicial sets, all cofibrations and trivial cofibrations are symmetrizable (Proposition 60). If $\mathscr{C}$ is the unstable model category of motivic spaces over a base, i.e. the model category for the unstable $\mathbb{A}^1$-homotopy category of schemes, cofibrations come up from the simplicial side, so that they are symmetrizable too. The $\mathbb{A}^1$-localization is a crux, and Theorem 42 gives that symmetrizability of trivial cofibrations is stable under $\mathbb{A}^1$-localization. In turn, this gives that trivial cofibrations between motivic spaces are symmetrizable, so that the above Theorem 22 and Theorem 25 are applicable in the motivic unstable homotopy theory of schemes over a base. Collecting all these things together we obtain the above Theorem A (Theorem 62 in the text).

In the stable world, the approach is different. In this paper, a stable homotopy category is the homotopy category of the category $\mathscr{S}$ of symmetric spectra over a closed symmetric monoidal model category $\mathscr{C}$, stabilizing a smash-with-T functor for a cofibrant object T in $\mathscr{C}$, see Sections 7 and 8 in [11]. This generalizes topological symmetric spectra introduced and studied in [12]. The symmetricity of spectra is essential to have the monoidal structure and its compatibility with the model one, see Theorem 8.11 in [11]. There are two crucial ingredients in working out symmetric powers of symmetric spectra. The first one is the existence and construction of the positive stable model structure for abstract symmetric spectra (Definition 47 and Theorem 50). The second ingredient is that n-th monoidal powers of positively cofibrant spectra are positively level-wise Σ_n-equivariantly cofibrant (Proposition 53). Using these results we prove (Theorem 55) a pretty general version of the theorem due to Elmendorf, Kriz, Mandell and May saying that the n-th symmetric power of a positively cofibrant topological spectrum is stably equivalent to the n-th homotopy symmetric power of that spectrum, see [3], Chapter III, Theorem 5.1, and [15], Lemma 15.5. Our method, however, is different from the one in [15]. In constructing positive model structures we systematically use Hirschhorn's localization and in proving Theorem 55 we use Theorem 7 on the stability of symmetrizable (trivial) cofibration under cofibrant generation. Theorem 55 implies that symmetric powers preserve positive and stable weak equivalences between cofibrant objects in the positive model structure in $\mathscr{S}$ (Corollary 56). In one turn, this gives λ-structure of left derived symmetric powers in the stable homotopy category $Ho(\mathscr{T})$ (Corollary 57). Notice also that the left derived symmetric powers of symmetric spectra are canonically isomorphic to the corresponding homotopy symmetric powers. Now, applying the above general results for symmetric spectra to motivic symmetric spectra of schemes over a base, we obtain Theorem B (Theorem 64 below).

2. Preliminary results

To get started we recall the notion of a closed symmetric monoidal model category $\mathscr{C}$. Such a category is equipped with three classes of morphisms, weak equivalences, fibrations and cofibrations, which have the standard lifting properties and meanings, see Chapter 1, §1 in [19], or Section 1.1 in [10]. The monoidality of $\mathscr{C}$ means that we have a functor $\wedge : \mathscr{C} \times \mathscr{C} \to \mathscr{C}$ sending any ordered pair of objects X, Y into their monoidal product $X \wedge Y$, and that product is symmetric, i.e. there exists a functorial transposition isomorphism $X \wedge Y \simeq Y \wedge X$. Moreover, the product $\wedge$ is also functorially associative, and there exists a unit object $\mathbb{1}$, such that $\mathbb{1} \wedge X \simeq X$ and $X \wedge \mathbb{1} \simeq X$ for any X in $\mathscr{C}$. The monoidal product could be also denoted by $\otimes$ but we prefer to keep to the "pointed" notation $\wedge$. Coproducts will be denoted by $\vee$.

A substantial thing here is that monoidality has to be compatible with the model structure. Namely, let $f : X \to Y$ and $f' : X' \to Y'$ be two morphisms in $\mathscr{C}$ and let

$$(X \wedge Y') \vee_{X \wedge X'} (Y \wedge X')$$

be the colimit of the diagram

$$\begin{tikzcd} X \wedge X' \arrow[r, "f \wedge \mathrm{id}"] \arrow[d, "\mathrm{id} \wedge f'"] & Y \wedge X' \\ X \wedge Y' & \end{tikzcd}$$

A pushout product of f and f' is, by definition, the unique map

$$f \Box f' : (X \wedge Y') \vee_{X \wedge X'} (Y \wedge X') \longrightarrow Y \wedge Y'$$

induced by the above colimit. The relation between the model and monoidal structures can be expressed by the following axioms, see Section 4.2 in [10]:

(A1) If f and f' are cofibrations then $f \square f'$ is also a cofibration. If, in addition, one of the maps f and f' is a weak equivalence, then so is $f \square f'$.
(A2) If $q : Q\mathbb{1} \to \mathbb{1}$ is a cofibrant replacement for the unit object $\mathbb{1}$, then the maps $q \wedge \mathrm{id} : Q\mathbb{1} \wedge X \to \mathbb{1} \wedge X$ and $\mathrm{id} \wedge q : X \wedge Q\mathbb{1} \to X \wedge \mathbb{1}$ are weak equivalences for all cofibrant X.

Here (A1) is called the pushout product axiom, and (A2) is called the unit axiom. The functor $X \wedge -$ has right adjoint functor $\underline{\mathrm{Hom}}(X, -)$. It follows that $X \wedge -$ commutes with colimits.

If $\mathscr{C}$ is simplicial, we will require that the simplicial structure is compatible with all the structures above in the sense of Definition 4.2.18 in [10].

Now, for any natural number n let Σ_n be the symmetric group of permutations of n elements, considered as a category with single object and morphisms being elements of the group. Given an object X in $\mathscr{C}$ we have a functor from Σ_n to $\mathscr{C}$ sending the unique object in Σ_n into $X^{\wedge n}$, and permuting factors using the commutativity and associativity constrains in $\mathscr{C}$. The n-th symmetric power $\mathrm{Sym}^n(X)$ of X is a colimit of this functor. Clearly, Sym^n is an endofunctor on $\mathscr{C}$.

Lemma 1. *Suppose that $\mathscr{C}$ is a closed symmetric monoidal model category. Assume, moreover, that $\mathscr{C}$ is a simplicial model category, and the functor $K \mapsto \mathbb{1} \wedge K$ from simplicial sets to $\mathscr{C}$ is symmetric monoidal. Let $f, g : X \rightrightarrows Y$ be two morphisms in $\mathscr{C}$ which are left homotopic, i.e. there exists a morphism $H : X \wedge \Delta[1] \to Y$, such that $H_0 = f$ and $H_1 = g$, where $\Delta[1]$ is the simplicial interval in $\Delta^{op}\mathscr{S}ets$. Then, for any natural n, the morphism $\mathrm{Sym}^n(f)$ is left homotopic to the morphism $\mathrm{Sym}^n(g)$.*

Proof. Let $\delta_n : \Delta[1] \to \Delta[1]^{\wedge n}$ be the diagonal morphism for the simplicial interval $\Delta[1]$, and let $\alpha_n : X^{\wedge n} \wedge \Delta[1] \to (X \wedge \Delta[1])^{\wedge n}$ be the composition of the morphism $\mathrm{id}_{X^{\wedge n}} \wedge \delta_n$ with the isomorphism between $X^{\wedge n} \wedge \Delta[1]^{\wedge n}$ and $(X \wedge \Delta[1])^{\wedge n}$. Then $H^{\wedge n} \circ \alpha_n : X^{\wedge n} \wedge \Delta[1] \to Y^{\wedge n}$ is a left homotopy between $f^{\wedge n}$ and $g^{\wedge n}$. The cylinder functor $- \wedge \Delta[1]$ has right adjoint, so commutes with colimits. Permuting factors does not affect the diagonal, and the functor $K \mapsto \mathbb{1} \wedge K$ is symmetric monoidal by the hypothesis. Therefore, the permutation of factors in $(X \wedge \Delta[1])^{\wedge n}$ is coherent with the permutation of factors in $X^{\wedge n}$ in the product $X^{\wedge n} \wedge \Delta[1]$. Taking colimits over Σ_n we obtain a left homotopy between $\mathrm{Sym}^n(f)$ and $\mathrm{Sym}^n(g)$. $\square$

Example 2. The existence of a simplicial structure, and its compatibility with the symmetric monoidal structure on $\mathscr{C}$ in Lemma 1 are essential. Indeed, let $\mathrm{Kom}(\mathbb{Z})$ be the category of unbounded complexes of abelian groups. The category $\mathrm{Kom}(\mathbb{Z})$ inherits the symmetric monoidal structure via total complexes $\mathrm{Tot}(- \otimes -)$, and has a natural structure of a model category whose weak equivalences are quasi-isomorphisms and fibrations are termwise epimorphisms, see Section 2.3 in [10]. Then $\mathrm{Kom}(\mathbb{Z})$ is a closed symmetric monoidal model category by Proposition 4.2.13 in [10]. It is not known whether $\mathrm{Kom}(\mathbb{Z})$ is a simplicial model category, see page 114 in [10]. The following argument, taken together with Lemma 1, shows that there is no a simplicial structure compatible with the monoidal model structure on $\mathrm{Kom}(\mathbb{Z})$, such that the functor $K \mapsto \mathbb{1} \wedge K$ would be symmetric monoidal. Let X be the complex $\ldots \to 0 \to \mathbb{Z} \stackrel{\mathrm{id}}{\to} \mathbb{Z} \to 0 \to \ldots$, where $\mathbb{Z}$ is concentrated in degrees -1 and 0 respectively. This complex is homotopically trivial. On the other hand, a calculation shows that $\mathrm{Sym}^2(X)$ is the complex

$$\ldots \to 0 \to \mathbb{Z}/2 \xrightarrow{0} \mathbb{Z} \xrightarrow{\mathrm{id}} \mathbb{Z} \to 0 \to \ldots\ ,$$

where $\mathbb{Z}/2$ stands in degree -2. Clearly, this $\mathrm{Sym}^2(X)$ has non-trivial cohomology group in degree -2.

Let now $\mathscr{C}$ be as in Lemma 1, and let $Ho(\mathscr{C})$ be the homotopy category of $\mathscr{C}$. A naive way to define symmetric powers in $Ho(\mathscr{C})$ would be through Lemma 1 and the standard treatment of homotopy categories as subcategories of fibrant–cofibrant objects factorized by left homotopies on Hom-sets, see [10, 1.2] or [19]. Indeed, let $\mathscr{C}_{\mathrm{cf}}$ be the full subcategory of objects which are fibrant and cofibrant simultaneously. Let, furthermore, $ho(\mathscr{C})$ be the quotient category of $\mathscr{C}_{\mathrm{cf}}$ by left homotopic morphisms between fibrant–cofibrant objects in $\mathscr{C}$. As symmetric powers respect left homotopies by Lemma 1, we have now a functor $\mathrm{Sym}^n : ho(\mathscr{C}) \to Ho(\mathscr{C})$. The category $\mathscr{C}$, being a model category, is endowed with a fibrant replacement functor $R : \mathscr{C} \to \mathscr{C}_{\mathrm{f}}$ and a cofibrant replacement functor $Q : \mathscr{C} \to \mathscr{C}_{\mathrm{c}}$. Combining both we obtain mixed replacement functors RQ and QR from $\mathscr{C}$ to the full subcategory $\mathscr{C}_{\mathrm{cf}}$ of fibrant–cofibrant objects in $\mathscr{C}$, any of which induces a quasi-inverse to the obvious functor from $ho(\mathscr{C})$ to $Ho(\mathscr{C})$. Then one might wish to construct an endofunctor Sym^n on $Ho(\mathscr{C})$ as a composition of this quasi-inverse and the above Sym^n. However, in general this method does not give left derived symmetric powers on $Ho(\mathscr{C})$.

3. Symmetrizable cofibrations

In this section, we introduce the notion of symmetrizable (trivial) cofibrations. The main result, Theorem 7, asserts that this property is stable under pushouts, retracts and transfinite compositions. This gives that, in order to check symmetrizability of (trivial) cofibrations, it is enough to examine it on generating (trivial) cofibrations, see Corollaries 9, 10 and 11.

Let $\mathscr{C}$ be a closed symmetric monoidal model category with the monoidal product $\wedge : \mathscr{C} \times \mathscr{C} \to \mathscr{C}$. For any two morphisms $f : X \to Y$ and $f' : X' \to Y'$ in $\mathscr{C}$, let

$$\Box(f, f') = (X \wedge Y') \vee_{X \wedge X'} (Y \wedge X')$$

be the coproduct over $X \wedge X'$ (not to be confused with the morphism $f \square f'$ defined in Section 2 before the axioms (A1) and (A2)). The pushout product $\square$ is commutative and associative in the obvious sense. For example, for any three morphisms $f : X \to Y$, $f' : X' \to Y'$ and $f'' : X'' \to Y''$ in $\mathscr{C}$ the morphism $(f \square f') \square f''$ is the same as the morphism $f \square (f' \square f'')$ up to the canonical isomorphism between $\square(f \square f', f'')$ and $\square(f, f' \square f'')$. Since $\square$ is an associative operation, for any finite collection $f_i : X_i \to Y_i$, $i = 1, \dots, l$, of morphisms in $\mathscr{C}$ we have a well defined morphism

$$f_1 \square \dots \square f_l : \square(f_1, \dots, f_l) \longrightarrow Y_1 \wedge \dots \wedge Y_l \ .$$

For simplicity, let $X' = X$, $Y' = Y$ and $f' = f$. Then we have the $\square$-squares $\square_1^2(f) = \square(f, f)$ and $f^{\square 2} = f \square f$, which can be generalized for higher degrees as follows. Let Γ be the category with two objects 0 and 1 and one morphism $0 \to 1$, and let Γ^n be the n-fold Cartesian product of Γ with itself. Objects in Γ^n are ordered n-tuples of 0's and 1's. A functor $K : \Gamma \to \mathscr{C}$ is just a morphism $f : X \to Y$ in $\mathscr{C}$. It is also natural to write $K(f)$ rather than K, since K is fully determined by the morphism f. For any natural n let K^n be the composition of the n-fold Cartesian product $\Gamma^n \to \mathscr{C}^n$ and the functor $\mathscr{C}^n \stackrel{\wedge}{\to} \mathscr{C}$. For any $0 \leq i \leq n$ one has a full subcategory Γ_i^n in Γ^n generated by n-tuples having not more than i units in them. The restriction of K^n on Γ_i^n will be denoted by $K_i^n(f)$, or simply by K_i^n when f is clear. In other words, K_i^n is a subdiagram in K^n having not more than i factors Y in each vertex. Let then

$$\square_i^n(f) = \operatorname{colim} K_i^n(f)$$

or simply

$$\square_i^n = \operatorname{colim} K_i^n \ .$$

Since $K_0^n = X^n$ and $K_n^n = K^n$, we have that $\square_0^n = X^n$ and $\square_n^n = Y^n$, respectively. As K_{i-1}^n is a subdiagram in K_i^n one has a morphism on colimits

$$\square_{i-1}^n \longrightarrow \square_i^n$$

for any $1 \leq i \leq n$.

Suppose $\mathscr{C}$ is cofibrantly generated. Let G be a finite group considered as a one-object category, and let $\mathscr{C}^G$ be the category of functors from G to $\mathscr{C}$. We shall be using the standard model structure on $\mathscr{C}^G$ provided by Theorem 11.6.1 in [8]. In particular, given a morphism f in $\mathscr{C}^G$, it is a weak equivalence (fibration) in $\mathscr{C}^G$ if and only if the same f, as a morphism in $\mathscr{C}$, is a weak equivalence (fibration) in $\mathscr{C}$. For any object X in $\mathscr{C}^G$, let X/G be the colimit of the action of the group G on X. This is a functor from $\mathscr{C}^G$ to $\mathscr{C}$ preserving cofibrations, see Theorem 11.6.8 in [8].

The group Σ_n acts on Γ^n and so on K^n. Each subcategory Γ_i^n is invariant under the action of Σ_n. Then Σ_n acts on K_i^n and so on $\square_i^n$. Let

$$\tilde{\Box}_i^n(f) = \operatorname{colim}_{\Sigma_n} \Box_i^n(f)$$

for each index i. Obviously, $\tilde{\Box}_0^n(f) = \mathrm{Sym}^n(X)$ and $\tilde{\Box}_n^n(f) = \mathrm{Sym}^n(Y)$, and for each index i we have a universal morphism between colimits

$$\tilde{\Box}_{i-1}^n(f) \longrightarrow \tilde{\Box}_i^n(f) \; .$$

Sometimes we will drop the morphism f from the notation writing

$$\tilde{\Box}_i^n = \operatorname{colim}_{\Sigma_n} \Box_i^n \; ,$$

$$\tilde{\Box}_{i-1}^n \longrightarrow \tilde{\Box}_{i-1}^n \; ,$$

etc.

In new notation, the axiom (A1) of a monoidal model category says, in particular, that for any cofibration $f : X \to Y$ in $\mathscr{C}$ the pushout product

$$f^{\Box 2} : \Box_1^2(f) \longrightarrow Y \wedge Y$$

is also a cofibration in $\mathscr{C}$. By associativity, it implies that the morphism

$$f^{\Box n} : \Box_{n-1}^n(f) \longrightarrow Y^{\wedge n}$$

is a cofibration in $\mathscr{C}$ for any natural n, not only for $n = 2$. It doesn't mean, of course, that the Σ_n-equivariant morphism $f^{\Box n}$ is a cofibration in $\mathscr{C}^{\Sigma_n}$.

Definition 3. A morphism $f : X \to Y$ in $\mathscr{C}$ is said to be a *symmetrizable* (trivial) cofibration if the corresponding morphism

$$f^{\tilde{\Box} n} : \tilde{\Box}_{n-1}^n(f) \longrightarrow \mathrm{Sym}^n(Y)$$

is a (trivial) cofibration for any integer $n \geq 1$.

A symmetrizable (trivial) cofibration f is itself a (trivial) cofibration because $\tilde{\Box}_0^1(f) \to \mathrm{Sym}^1(Y)$ is nothing but the original morphism f.

Remark 4. If $f : X \to Y$ is a symmetrizable (trivial) cofibration in $\mathscr{C}$, it is not necessarily true that the Σ_n-equivariant morphism $f^{\Box n}$ is a cofibration in $\mathscr{C}^{\Sigma_n}$. Theoretically, it would also make sense to say that f is a *strongly symmetrizable* (trivial) cofibration if $f^{\Box n}$ is a (trivial) cofibration in $\mathscr{C}^{\Sigma_n}$. However, such defined strongly symmetrizable cofibrations are not of much use to us because, as the following example shows, they do not occur even in topology.

Example 5. Let $\mathscr{C}$ be the model category of simplicial sets $\Delta^{op}\mathscr{S}ets$. According to our notation, $\wedge$ in this $\mathscr{C}$ stands for the usual Cartesian product of simplicial sets. Let $E\Sigma_n$

be the contractible simplicial set with $(E\Sigma_n)_i = \Sigma_n^{\times i}$, and the diagonal action of Σ_n. The morphism $f : \emptyset \to X$ is a cofibration for any simplicial set X. Then the morphism $f^{\tilde{\square} n}$ from $\emptyset$ to $\mathrm{Sym}^n(X)$ is also a cofibration, for any $n \geq 1$. Hence, f is a symmetrizable cofibration. Similarly, one can show that all cofibrations in $\Delta^{op}\mathscr{S}ets$ are symmetrizable. On the other hand, the morphism $f^{\square n}$ from $\emptyset$ to $X^{\wedge n}$ is not a cofibration in $\mathscr{C}^{\Sigma_n}$. The reason is that the diagonal map from X to $X^{\wedge n}$ is Σ_n-equivariant. This has the effect that there are no Σ_n-morphisms from $X^{\wedge n}$ to $E\Sigma_n \wedge X^{\wedge n}$, as Σ_n acts term-wise freely on the simplicial set $E\Sigma_n \wedge X^{\wedge n}$. It follows that the morphism $f^{\square n}$ does not have a Σ_n-left lifting property with respect to the trivial fibration $E\Sigma_n \wedge X^{\wedge n} \to X^{\wedge n}$ in $\mathscr{C}^{\Sigma_n}$ and the identity map from $X^{\wedge n}$ to itself. Thus, f is not strongly symmetrizable in the sense of Remark 4.

Symmetrizability of (trivial) cofibrations is not always the case too. Example 2 shows that trivial cofibrations are not symmetrizable in the category $\mathrm{Kom}(\mathbb{Z})$. More importantly, cofibrations are not symmetrizable for symmetric spectra over simplicial sets, see Remark 58 below. This is why we shall give one more definition of (strong) symmetrizability, which will serve all the needs relevant to symmetric powers of symmetric spectra.

Let $\mathscr{C}$ be a closed symmetric monoidal model category, let $\mathscr{D}$ be a cofibrantly generated model category, and let $F : \mathscr{C} \to \mathscr{D}$ be a functor from $\mathscr{C}$ to $\mathscr{D}$. Then F induces a functor from $\mathscr{C}^G$ to $\mathscr{D}^G$, which will be denoted by the same symbol F. A finite collection $\{n_1, \dots, n_l\}$ of non-negative integers will be called a multidegree.

Definition 6. A class of morphisms M in $\mathscr{C}$ will be called a *symmetrizable class* of (trivial) F-cofibrations in $\mathscr{C}$ if for any finite collection $\{f_1, \dots, f_l\}$ of morphisms in the class M and any multidegree $\{n_1, \dots, n_l\}$ the morphism

$$F(f_1^{\tilde{\square} n_1} \square \dots \square f_l^{\tilde{\square} n_l})$$

is a (trivial) cofibration in the model category $\mathscr{D}$. The class M will be called a *strongly symmetrizable class* of (trivial) F-cofibrations in $\mathscr{C}$ if for any finite collection $\{f_1, \dots, f_l\}$ of morphisms in M and any multidegree $\{n_1, \dots, n_l\}$ the morphism

$$F(f_1^{\square n_1} \square \dots \square f_l^{\square n_l})$$

is a (trivial) cofibration in the model category $\mathscr{D}^{\Sigma_{n_1} \times \dots \times \Sigma_{n_l}}$.

Notice that if $\mathscr{D} = \mathscr{C}$ and F is the identity functor, then M is a (strongly) symmetrizable class of (trivial) Id-cofibrations if and only if M consists of (strongly) symmetrizable (trivial) cofibrations in $\mathscr{C}$. The case $l > 1$ is essential when F is not monoidal. This will hold in the applications to symmetric spectra in Section 9.

Let now λ be an ordinal and let X be a functor from λ to a model category $\mathscr{C}$ preserving colimits (although λ is not necessarily cocomplete). To shorten notation, for any ordinal $\alpha < \lambda$ let X_α be the object $X(\alpha)$, and for any two ordinals α and β, such that $\alpha \leq \beta < \lambda$, let $f_{\beta,\alpha} = X(\alpha \leq \beta)$. Let also $X_\infty = \mathrm{colim}\,(X)$ and, for any ordinal $\alpha < \lambda$, let $f_{\infty,\alpha} : X_\alpha \to X_\infty$ be the canonical morphism into colimit. Since the set of objects in λ has the minimal object 0, we have the canonical morphism $f_\infty : X_0 \to X_\infty$, which is called a transfinite composition induced by the functor X.

Theorem 7. *Let $\mathscr{C}$ be a cofibrantly generated closed symmetric monoidal model category, $F : \mathscr{C} \to \mathscr{D}$ a functor from $\mathscr{C}$ to a cofibrantly generated model category $\mathscr{D}$ commuting with colimits, and let M be a (strongly) symmetrizable class of (trivial) F-cofibrations in $\mathscr{C}$. Let ϕ be a morphism of one of the following types:*

(A) *a pushout of a morphism from M;*
(B) *a retract of a morphism from M;*
(C) *a composition $g \circ f$, where f and g are two composable morphisms from M;*
(D) *a transfinite composition $f_\infty : X_0 \to X_\infty$ induced by a functor $X : \lambda \to \mathscr{C}$, where λ is an ordinal, X commutes with colimits, and for any ordinal $\alpha < \lambda$, such that $\alpha + 1 < \lambda$, the morphism $f_{\alpha+1,\alpha} : X_\alpha \to X_{\alpha+1}$ is in M.*

Then the class $M \cup \{\phi\}$ is a (strongly) symmetrizable class of (trivial) F-cofibrations in $\mathscr{C}$ too.

Remark 8. Item (C) can be considered as a particular case of item (D). The category $\mathscr{D}$ is required to be cofibrantly generated merely to have a model structure on the category $\mathscr{D}^{\Sigma_n}$.

The proof of Theorem 7 occupies the next section of the paper. Now we discuss its consequences. Suppose $\mathscr{C}$ is cofibrantly generated by a set of generating cofibrations I and a set of generating trivial cofibrations J.

Corollary 9. *If I is a (strongly) symmetrizable set of F-cofibrations, then the class of all cofibrations in $\mathscr{C}$ is a (strongly) symmetrizable class of F-cofibrations. Similarly, if J is a (strongly) symmetrizable set of trivial F-cofibrations, then the class of all trivial cofibrations in $\mathscr{C}$ is a (strongly) symmetrizable class of trivial F-cofibrations.*

Proof. By Theorem 7, the class of retracts of relative I-cell complexes is a (strongly) symmetrizable class of F-cofibrations. On the other hand, this class coincides with all cofibrations in $\mathscr{C}$. Similar argument applies to trivial cofibrations. $\square$

Applying Corollary 9 to a cofibration $\emptyset \to X$ we obtain two more corollaries.

Corollary 10. *Suppose all morphisms in I are symmetrizable. Then any symmetric power $\mathrm{Sym}^n(X)$ of a cofibrant object X in $\mathscr{C}$ is cofibrant.*

Corollary 11. *If I is a strongly symmetrizable set of F-cofibrations, then for any cofibrant object X in $\mathscr{C}$ we have that $F(X^{\wedge n})$ is a cofibrant object in $\mathscr{D}^{\Sigma_n}$.*

For short, by abuse of notation, throughout the text we will say that I is symmetrizable if it consists of symmetrizable cofibrations, and that J is symmetrizable if it consists of symmetrizable trivial cofibrations.

Finally, we compare the pointed v.s. unpointed cases of our setup. Assuming the terminal object $*$ is the monoidal unit and cofibrant in $\mathscr{C}$, the pointed category $\mathscr{C}_* = * \downarrow \mathscr{C}$ inherits the monoidal model structure by Proposition 4.2.9 in [10].

Lemma 12. *Let f be a morphism in $\mathscr{C}_*$, which is a symmetrizable (trivial) cofibration as a morphism in $\mathscr{C}$. Then f is a symmetrizable (trivial) cofibration as a morphism in $\mathscr{C}_*$.*

Proof. This follows from the fact that $f^{\Box n}$ in $\mathscr{C}_*$ is a pushout of $f^{\Box n}$ in $\mathscr{C}$. $\square$

4. The proof of Theorem 7

First we collect some technical lemmas needed in proving the theorem. If $f : X \to Y$ is a morphism in $\mathscr{C}$ and there exists a pushout square

$$\begin{tikzcd} X' \arrow[r] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r] & Y \end{tikzcd}$$

then sometimes we will write

$$f = \mathrm{psht}(f') \ ,$$

not specifying the horizontal morphisms of the square.

Lemma 13. *Let $f = \mathrm{psht}(f')$, $e : A \to B$ a morphism in $\mathscr{C}$ and let*

$$d : \Box(f', e) \longrightarrow \Box(f, e)$$

be the universal morphism between two colimits induced by the pushout square above. Then the commutative square

$$\begin{tikzcd}
\square(f',e) \arrow[r, "d"] \arrow[d, "f'\square e"'] & \square(f,e) \arrow[d, "f\square e"] \\
Y'\wedge B \arrow[r] & Y\wedge B
\end{tikzcd}$$

is pushout, i.e. $\mathrm{psht}(f')\square e = \mathrm{psht}(f'\square e)$.

Proof. As $\wedge$-multiplication is a left adjoint, and so it commutes with colimits, the commutative squares

$$\begin{tikzcd}
X'\wedge A \arrow[r] \arrow[d, "f'\wedge\mathrm{id}"'] & X\wedge A \arrow[d, "f\wedge\mathrm{id}"] \\
Y'\wedge A \arrow[r] & Y\wedge A
\end{tikzcd}
\qquad
\begin{tikzcd}
X'\wedge B \arrow[r] \arrow[d, "f'\wedge\mathrm{id}"'] & X\wedge B \arrow[d, "f\wedge\mathrm{id}"] \\
Y'\wedge B \arrow[r] & Y\wedge B
\end{tikzcd}$$

are pushout. The morphism e induces a morphism from the left pushout square to the right one. This and the universal property of the colimits $\square(f',e)$ and $\square(f,e)$ allow to show that the commutative square in question is pushout. $\square$

Lemma 14. *Let $f_1,\dots,f_n$ be a collection of morphisms in $\mathscr{C}$. Then we have*

$$\mathrm{psht}(f_1)\square\dots\square\mathrm{psht}(f_n) = \mathrm{psht}(f_1\square\dots\square f_n)\ .$$

Proof. Use Lemma 13 and associativity of the $\square$-product. $\square$

Let G be a finite group and let H be a subgroup in it. The natural restriction $\mathrm{res}^G_H : \mathscr{C}^G \to \mathscr{C}^H$ has left adjoint functor $\mathrm{cor}^G_H : \mathscr{C}^H \to \mathscr{C}^G$, such that $(\mathrm{cor}^G_H, \mathrm{res}^G_H)$ is a Quillen adjunction, see Theorem 11.9.4 in [8]. Recall that $\mathrm{cor}^G_H(X) \simeq (G\times X)/H$ and $\mathrm{cor}^G_H(X)/G \simeq X/H$.

Lemma 15. *Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be two composable morphisms in $\mathscr{C}$, and let n be a positive integer. Then the morphism $(gf)^{\square n} : \square^n_{n-1}(gf) \to Z^{\wedge n}$ is a composition*

$$g^{\square n}\circ \mathrm{psht}(\mathrm{cor}^{\Sigma_n}_{\Sigma_{n-1}\times\Sigma_1}(g^{\square(n-1)}\square f))\circ\dots\circ \mathrm{psht}(\mathrm{cor}^{\Sigma_n}_{\Sigma_1\times\Sigma_{n-1}}(g\square f^{\square(n-1)}))\circ \mathrm{psht}(f^{\square n})\ ,$$

where $\Sigma_i\times\Sigma_{n-i}$ is canonically embedded into Σ_n for each i, and $\mathrm{psht}(f^{\square n})$ is a pushout of $f^{\square n}$ with respect to the universal morphism $\square^n_{n-1}(f) \to \square^n_{n-1}(gf)$.

Proof. Let J be the category $0 \to 1 \to 2$. A pair of subsequent morphisms $f : X \to Y$ and $g : Y \to Z$ in $\mathscr{C}$ can be considered as a functor $K(f,g) : J \to \mathscr{C}$ from J to $\mathscr{C}$. Let J^n and $\mathscr{C}^n$ be the Cartesian n-th powers of the categories J and $\mathscr{C}$ respectively, and let $K^n(f,g) : J^n \to \mathscr{C}$ be the composition of the n-th Cartesian power of the functor $K(f,g)$ and the n-th monoidal product $\wedge : \mathscr{C}^n \to \mathscr{C}$. In particular, $K^n(f,g)$ is a commutative diagram in $\mathscr{C}$, whose vertices are monoidal products of the three objects X, Y and Z. Notice that the order of the factors is important here.

For short, let $K^n = K^n(f,g)$, and consider a subdiagram L in K^n generated by the vertices containing at least one factor X, and for any index $i \in \{0, 1, \dots, n\}$ let K_i^n be a subdiagram in K^n generated by vertices containing $\leq i$ factors Z. Let also $L_i = L \cup K_i^n$ and put $L_{-1} = L$. Then we have a filtration

$$L_{-1} \subset L_0 \subset L_1 \subset L_2 \subset \cdots \subset L_n = K^n$$

and, correspondingly, a chain of morphisms between colimits

$$\operatorname{colim}(L_{-1}) \to \operatorname{colim}(L_0) \to \operatorname{colim}(L_1) \to \cdots \to \operatorname{colim}(K^n)\ ,$$

whose composition is nothing but $(gf)^{\square n}$.

For any $0 \leq i \leq n$ the object $\square(g^{\square i}, f^{\square(n-i)})$ is a colimit of a subdiagram in L_{i-1}, so that one has a universal morphism from $\square(g^{\square i}, f^{\square(n-i)})$ to $\operatorname{colim}(L_{i-1})$. Since $Z^{\wedge i} \wedge Y^{\wedge(n-i)}$ is a vertex in the diagram L_i, we have a morphism from $Z^{\wedge i} \wedge Y^{\wedge(n-i)}$ to $\operatorname{colim}(L_i)$. Finally, we have a standard morphism $g^{\square i} \square f^{\square(n-i)} : \square(g^{\square i}, f^{\square(n-i)}) \to Z^{\wedge i} \wedge Y^{\wedge(n-i)}$. Collecting these morphisms together we get a commutative diagram

$$\begin{tikzcd}
\square(g^{\square i}, f^{\square(n-i)}) \arrow[r] \arrow[d] & Z^{\wedge i} \wedge Y^{\wedge(n-i)} \arrow[d] \\
\operatorname{colim}(L_{i-1}) \arrow[r] & \operatorname{colim}(L_i)
\end{tikzcd}$$

This is a $\Sigma_i \times \Sigma_{n-i}$-equivariant commutative diagram, which yields a Σ_n-equivariant commutative diagram

$$\begin{tikzcd}
\operatorname{cor}_{\Sigma_i \times \Sigma_{n-i}}^{\Sigma_n}(\square(g^{\square i}, f^{\square(n-i)})) \arrow[r] \arrow[d] & \operatorname{cor}_{\Sigma_i \times \Sigma_{n-i}}^{\Sigma_n}(Z^{\wedge i} \wedge Y^{\wedge(n-i)}) \arrow[d] \\
\operatorname{colim}(L_{i-1}) \arrow[r] & \operatorname{colim}(L_i)
\end{tikzcd}$$

Straightforward verification shows that this is a pushout square. $\square$

Lemma 16. *For any three morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$ and $A \xrightarrow{e} B$ in $\mathscr{C}$ we have that*

$$(gf)\square e = (g\square e) \circ \varkappa \, ,$$

where $\varkappa : \square(gf, e) \to \square(g, e)$ is a universal morphism between colimits and the square

$$\begin{tikzcd}
\square(f,e) \arrow[r] \arrow[d, "f\square e"'] & \square(gf,e) \arrow[d, "\varkappa"] \\
Y \wedge V \arrow[r] & \square(g,e)
\end{tikzcd}$$

is pushout, i.e. we have $(gf)\square e = (g\square e) \circ \mathrm{psht}(f\square e)$.

Proof. The top horizontal morphism $\square(f, e) \to \square(gf, e)$ in the above diagram is also a universal morphism between colimits. The proof of the lemma then follows from the appropriate commutative diagrams for the products $f\square e$, $gf\square e$ and $g\square e$ involved into the lemma. $\square$

Lemma 17. *Let $X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \cdots \xrightarrow{f_n} X_{n+1}$ and $e : A \to B$ be morphisms in $\mathscr{C}$. Then one has*

$$(f_n \circ \cdots \circ f_1)\square e = (f_n\square e) \circ \mathrm{psht}(f_{n-1}\square e) \circ \cdots \circ \mathrm{psht}(f_1\square e) \, .$$

Proof. Use induction by n. If $n = 2$ then the lemma is just Lemma 16. For the inductive step,

$$(f_n \circ \cdots \circ f_1)\square h = (f_n \circ (f_{n-1} \circ \cdots \circ f_1))\square h =$$
$$(f_n\square h) \circ \mathrm{psht}((f_{n-1} \circ \cdots \circ f_1)\square h) \, ,$$

where the last equality is provided by Lemma 16 too. $\square$

Lemma 18. *Let G and G' be two finite groups, let H be a subgroup in G and H' be a subgroup in G'. Let also $f : X \to Y$ and $f' : X' \to Y'$ be two morphisms in $\mathscr{C}$. Then*

$$\mathrm{cor}_H^G(f)\square \mathrm{cor}_{H'}^{G'}(f') = \mathrm{cor}_{H\times H'}^{G\times G'}(f\square f') \, .$$

Proof. The lemma holds true because $\wedge$-multiplication commutes with colimits and the order of counting colimits is not important. $\square$

Lemma 19. *Let λ be an ordinal. For any two functors X and Y from λ to $\mathscr{C}$, not necessarily preserving colimits, one has a canonical isomorphism*

$$\mathrm{colim}_{\alpha<\lambda}(X_\alpha \wedge Y_\alpha) \simeq (\mathrm{colim}_{\alpha<\lambda} X_\alpha) \wedge (\mathrm{colim}_{\beta<\lambda} Y_\beta) .$$

Proof. Indeed, as the monoidal product $\wedge$ in $\mathscr{C}$ is closed, smashing with an object commutes with colimits. Therefore, we have two canonical isomorphisms

$$(\mathrm{colim}_{\alpha<\lambda} X_\alpha) \wedge (\mathrm{colim}_{\beta<\lambda} Y_\beta) \simeq \mathrm{colim}_{\alpha<\lambda}(X_\alpha \wedge \mathrm{colim}_{\beta<\lambda} Y_\beta) \simeq$$
$$\simeq \mathrm{colim}_{\alpha<\lambda}\mathrm{colim}_{\beta<\lambda}(X_\alpha \wedge Y_\beta) .$$

Since all arrows in the diagram $X \wedge Y$ are targeted towards the diagonal objects $X_\alpha \wedge Y_\alpha$, the last colimit is canonically isomorphic to the colimit of these objects. $\square$

Let now λ be an ordinal and let X be a colimit-preserving functor from the ordinal λ to the category $\mathscr{C}$. For any two ordinals α and β, such that $\alpha \leq \beta < \lambda$, let $\square^n_{n-1}(f_{\alpha,0}) \to \square^n_{n-1}(f_{\beta,0})$ be the universal morphism from Lemma 15 being applied to the composition $f_{\beta,0} = f_{\beta,\alpha} \circ f_{\alpha,0}$. Similarly, let $\square^n_{n-1}(f_{\alpha,0}) \to \square^n_{n-1}(f_\infty)$ be the universal morphism from Lemma 15 applied to the composition $f_\infty = f_{\infty,\alpha} \circ f_{\alpha,0}$. It is not hard to verify that the collection of objects $\square^n_{n-1}(f_{\alpha,0})$ and morphisms $\square^n_{n-1}(f_{\alpha,0}) \to \square^n_{n-1}(f_{\beta,0})$ gives a functor from λ to $\mathscr{C}$.

Lemma 20. *In the above terms, there are canonical isomorphisms*

$$\square^n_{n-1}(f_\infty) \simeq \mathrm{colim}_{\alpha<\lambda}\square^n_{n-1}(f_{\alpha,0}) ,$$
$$X_\infty^{\wedge n} \simeq \mathrm{colim}_{\alpha<\lambda} X_\alpha^{\wedge n} ,$$

and the square

$$\begin{tikzcd}
\square^n_{n-1}(f_{\alpha,0}) \arrow[r] \arrow[d, "f^{\square n}_{\alpha,0}"'] & \square^n_{n-1}(f_\infty) \arrow[d, "f^{\square n}_\infty"] \\
X_\alpha^{\wedge n} \arrow[r] & X_\infty^{\wedge n}
\end{tikzcd}$$

is commutative for any α, i.e.

$$f^{\square n}_\infty = \mathrm{colim}_{\alpha<\lambda}(f^{\square n}_{\alpha,0}) .$$

Proof. By Lemma 19,

$$K^n_i(f_\infty) \simeq \mathrm{colim}_{\alpha<\lambda} K^n_i(f_{\alpha,0})$$

for any index i, where the colimit is taken in the category of functors from subcategories in I^n to $\mathscr{C}$. It implies the following computation:

$$\Box_i^n(f_\infty) \simeq \operatorname{colim} K_i^n(f_\infty) \simeq \operatorname{colim}\left(\operatorname{colim}_{\alpha<\lambda} K_i^n(f_{\alpha,0})\right) \simeq$$
$$\simeq \operatorname{colim}_{\alpha<\lambda}\left(\operatorname{colim} K_i^n(f_{\alpha,0})\right) \simeq \operatorname{colim}_{\alpha<\lambda} \Box_i^n(f_{\alpha,0}) \ .$$

In particular,

$$\Box_{n-1}^n(f_\infty) \simeq \operatorname{colim}_{\alpha<\lambda} \Box_{n-1}^n(f_{\alpha,0}) \ ,$$
$$X_\infty^n \simeq \Box_n^n(f_\infty) \simeq \operatorname{colim}_{\alpha<\lambda} \Box_n^n(f_{\alpha,0}) \simeq \operatorname{colim}_{\alpha<\lambda} X_\alpha^n \ ,$$

and both isomorphisms are connected by the corresponding commutative square. $\Box$

Lemma 21. *Let $\mathscr{E}$ be a model category and let λ be an ordinal. Let*

$$U, V : \lambda \rightrightarrows \mathscr{E}$$

be two functors from λ to $\mathscr{E}$, both commuting with colimits, and let

$$\psi : U \to V$$

be a natural transformation. For any ordinal $\alpha < \lambda$, such that $\alpha + 1 < \lambda$, let

$$\begin{tikzcd} U_\alpha \arrow[r] \arrow[d, "\psi_\alpha"'] & U_{\alpha+1} \arrow[d] \\ V_\alpha \arrow[r] & W_\alpha \end{tikzcd}$$

be a pushout square, and let h_α be a universal morphism from the colimit W_α to $V_{\alpha+1}$. Assume that for any $\alpha < \lambda$, such that $\alpha + 1 < \lambda$, the morphism h_α and the morphism ψ_0 are cofibrations in $\mathscr{E}$. Then the universal morphism

$$\operatorname{colim}(\psi) : \operatorname{colim}(U) \to \operatorname{colim}(V)$$

is also a cofibration in $\mathscr{E}$.

Proof. For any ordinal $\alpha < \lambda$, such that $\alpha + 1 < \lambda$, let D_α be the diagram

$$
\begin{tikzcd}
U_0 \arrow[r] \arrow[d] & U_1 \arrow[r] \arrow[d] & \dots \arrow[r] & U_\alpha \arrow[r] \arrow[d] & U_{\alpha+1} \arrow[r] & \dots \arrow[r] & \mathrm{colim}\,(U) \\
V_0 \arrow[r] & V_1 \arrow[r] & \dots \arrow[r] & V_\alpha & & &
\end{tikzcd}
$$

Let also D_{-1} be the diagram

$$U_0 \to U_1 \to \cdots \to U_\alpha \to \cdots \to \mathrm{colim}\,(U)\ ,$$

and let D_λ be the diagram

$$
\begin{tikzcd}
U_0 \arrow[r] \arrow[d] & U_1 \arrow[r] \arrow[d] & \dots \arrow[r] & U_\alpha \arrow[r] \arrow[d] & U_{\alpha+1} \arrow[r] \arrow[d] & \dots \arrow[r] & \mathrm{colim}\,(U) \arrow[d] \\
V_0 \arrow[r] & V_1 \arrow[r] & \dots \arrow[r] & V_\alpha \arrow[r] & V_{\alpha+1} \arrow[r] & \dots \arrow[r] & \mathrm{colim}\,(V)
\end{tikzcd}
$$

Let now $S_\alpha = \mathrm{colim}\,(D_\alpha)$, $S_{-1} = \mathrm{colim}\,(D_{-1}) = \mathrm{colim}\,(U)$ and $S_\lambda = \mathrm{colim}\,(D_\lambda) = \mathrm{colim}\,(V)$. One has a transfinite filtration of diagrams

$$D_{-1} \subset D_0 \subset D_1 \subset \cdots \subset D_\alpha \subset D_{\alpha+1} \subset \cdots \subset D_\lambda\ .$$

Consequently, we obtain a decomposition of the morphism $\mathrm{colim}\,(\psi)$ into a transfinite composition

$$\mathrm{colim}\,(U) = S_{-1} \to S_0 \to S_1 \to \cdots \to S_\alpha \to S_{\alpha+1} \to \cdots \to S_\lambda = \mathrm{colim}\,(V)\ .$$

For any $\alpha < \lambda$, such that $\alpha + 1 < \lambda$, the square

$$
\begin{tikzcd}
W_\alpha \arrow[r, "h_\alpha"] \arrow[d] & V_{\alpha+1} \arrow[d] \\
S_\alpha \arrow[r] & S_{\alpha+1}
\end{tikzcd}
$$

is pushout. Since our input is that all h_α and ψ_0 are cofibrations, we get that the morphism $\mathrm{colim}\,(\psi)$ is a transfinite composition of cofibrations in $\mathscr{E}$. Since a transfinite composition of cofibrations is a cofibration, the lemma is proved. $\square$

Now we are ready to give the proof of Theorem 7. We will only consider the strong symmetrizability case. The symmetrizability assertion then follows by applying in addition the colimit under the action of the symmetric group.

Let $f', f_2, \dots, f_l$ be l morphisms in M and let f be a pushout of f'. To prove (A) we need to show that the morphism $F(f^{\square n} \square f_2^{\square n_2} \square \dots \square f_l^{\square n_l})$ is a cofibration in the category $\mathscr{D}^{\Sigma_n \times \Sigma_{n_2} \times \cdots \times \Sigma_{n_l}}$ for any multidegree $\{n, n_2, \dots, n_l\}$.

By Lemma 14, $f^{\square n} \square f_2^{\square n_2} \square \dots \square f_l^{\square n_l}$ is a pushout of $f'^{\,\square n} \square f_2^{\square n_1} \square \dots \square f_l^{\square n_l}$. Since F commutes with colimits, the morphism $F(f^{\square n} \square f_2^{\square n_2} \square \dots \square f_l^{\square n_l})$ is a pushout of $F(f'^{\,\square n} \square f_2^{\square n_1} \square \dots \square f_l^{\square n_l})$. Since the latest morphism is a cofibration in $\mathscr{D}^{\Sigma_n \times \Sigma_{n_2} \times \cdots \times \Sigma_{n_l}}$, the morphism $F(f^{\square n} \square f_2^{\square n_2} \square \dots \square f_l^{\square n_l})$ is a cofibration too. So, (A) is done.

To prove (B) we just notice that a retract of a cofibration is a cofibration, and retraction is a categoric property coomuting with colimits. This gives (B).

Let $f, g, f_2, \dots, f_l$ be $l+1$ morphisms in M, where f and g are composable. To prove (C) we need to show that for any multidegree $\{n, n_2, \dots, n_l\}$ the morphism $F((gf)^{\square n} \square f_2^{\square n_2} \square \dots \square f_l^{\square n_l})$ is a cofibration in $\mathscr{D}^{\Sigma_n \times \Sigma_{n_2} \times \cdots \times \Sigma_{n_l}}$.

By Lemma 15 we have that $(gf)^{\square n}$ is a composition of pushouts of the morphisms $\mathrm{cor}_{\Sigma_i \times \Sigma_{n-i}}^{\Sigma_n}(g^{\square i} \square f^{\square (n-i)})$ for $i = 0, 1, \dots, n$. By Lemma 17 and Lemma 14, the morphism $(gf)^{\square n} \square f_2^{\square n_2} \square \dots \square f_l^{\square n_l}$ is a composition of pushouts of the morphisms

$$\mathrm{cor}_{\Sigma_i \times \Sigma_{n-i}}^{\Sigma_n}(g^{\square i} \square f^{\square (n-i)}) \square f_2^{\square n_2} \square \dots \square f_l^{\square n_l} \ .$$

By Lemma 18, the latest morphism can be also viewed as the morphism

$$\mathrm{cor}_{\Sigma_i \times \Sigma_{n-i} \times \Sigma_{n_2} \times \cdots \times \Sigma_{n_l}}^{\Sigma_n \times \Sigma_{n_2} \times \cdots \times \Sigma_{n_l}}(g^{\square i} \square f^{\square (n-i)} \square f_2^{\square n_2} \square \dots \square f_l^{\square n_l}) \ .$$

Since any cor is a colimit and the functor F commutes with colimits, the morphism $F((gf)^{\square n} \square f_2^{\square n_2} \square \dots \square f_l^{\square n_l})$ is a composition of pushouts of morphisms of type

$$\mathrm{cor}_{\Sigma_i \times \Sigma_{n-i} \times \Sigma_{n_2} \times \cdots \times \Sigma_{n_l}}^{\Sigma_n \times \Sigma_{n_2} \times \cdots \times \Sigma_{n_l}}(F(g^{\square i} \square f^{\square (n-i)} \square f_2^{\square n_2} \square \dots \square f_l^{\square n_l})) \ .$$

Since the morphisms $f, g, f_2, \dots, f_l$ are taken from the class M, every morphism $F(g^{\square i} \square f^{\square (n-i)} \square f_2^{\square n_2} \square \dots \square f_l^{\square n_l})$ is a cofibration in the category $\mathscr{D}^{\Sigma_{n-i} \times \Sigma_i \times \Sigma_{n_2} \times \cdots \times \Sigma_{n_l}}$. As cor_H^G is a left Quillen functor for any group G and a subgroup H in it, we obtain that $F((gf)^{\square n} \square f_2^{\square n_2} \square \dots \square f_l^{\square n_l})$ is a cofibration in the category $\mathscr{D}^{\Sigma_n \times \Sigma_{n_2} \times \cdots \times \Sigma_{n_l}}$.

Now we prove (D). For any ordinal λ let $\mathrm{D}(\lambda)$ be the property (D) in the statement of the theorem being considered for this ordinal λ. We need to show $\mathrm{D}(\lambda)$ for any ordinal λ. To do that we are going to apply the method of transfinite induction. Namely, suppose that for any ordinal $\alpha < \lambda$ the property $\mathrm{D}(\alpha)$ is satisfied. We will show that this assumption implies that $\mathrm{D}(\lambda)$ holds true.

Consider a finite collection $f_2, \dots, f_l$ of morphisms in M. We need to show that for any positive integers $n, n_2, \dots, n_l$ the morphism $F(f_\infty^{\square n} \square f_2^{\square n_2} \square \dots \square f_l^{\square n_l})$ is a cofibration in the category $\mathscr{D}^{\Sigma_n \times \Sigma_{n_2} \times \dots \times \Sigma_{n_l}}$. If, for short, we denote the morphism $f_2^{\square n_2} \square \dots \square f_l^{\square n_l}$ by $e : A \to B$ then we need to show that for any positive integer n the morphism $F(f_\infty^{\square n} \square e)$ is a cofibration in $\mathscr{D}^{\Sigma_n \times \Sigma_{n_2} \times \dots \times \Sigma_{n_l}}$.

Our strategy is to apply Lemma 21 to the category $\mathscr{E} = \mathscr{D}^{\Sigma_n \times \Sigma_{n_2} \times \dots \times \Sigma_{n_l}}$, the functors $U = F(\square(f_{\alpha,0}^{\square n}, e))$, $V = F(X_\alpha^{\wedge n} \wedge B)$, and the natural transformation $\psi = F(f_{\alpha,0}^{\square n} \square e)$. First we show that colim (ψ) is nothing but the morphism $F(f_\infty^{\square n} \square e)$. This is provided by Lemma 20, which says that $f_\infty^{\square n} = \operatorname{colim}(f_{\alpha,0}^{\square n})$, the commutativity of the functor F with colimits, and the obvious fact that the right $\square$-multiplication is colimit-commutative too:

$$\operatorname{colim}(\psi) \simeq \operatorname{colim} F(f_{\alpha,0}^{\square n} \square e) \simeq$$
$$\simeq F(\operatorname{colim}(f_{\alpha,0}^{\square n} \square e)) \simeq F(\operatorname{colim}(f_{\alpha,0}^{\square n}) \square e) \simeq F(f_\infty^{\square n} \square e) \ .$$

Next, we have that

$$\psi_0 = F(f_{0,0}^{\square n} \square e) = F(\mathrm{id}_{X^{\wedge n}} \square e) = F(\mathrm{id}_{X^{\wedge n} \wedge B}) = \mathrm{id}_{F(X^{\wedge n} \wedge B)}$$

is a cofibration in $\mathscr{D}^{\Sigma_n \times \Sigma_{n_2} \times \dots \times \Sigma_{n_l}}$. In order to apply Lemma 21 it remains only to show that the universal morphisms h_α are cofibrations in $\mathscr{D}^{\Sigma_n \times \Sigma_{n_2} \times \dots \times \Sigma_{n_l}}$. We give an explicit description of h_α.

Let r_α be a pushout of the morphism $f_{\alpha,0}^{\square n}$ with respect to the universal morphism between colimits $\square_{n-1}^n(f_{\alpha,0}) \to \square_{n-1}^n(f_{\alpha+1,0})$. Applying Lemma 13 to the corresponding pushout square and the morphism $e : A \to B$ we get a pushout square

$$\begin{tikzcd}
\square(f_{\alpha,0}^{\square n}, e) \arrow[r] \arrow[d, "f_{\alpha,0}^{\square n} \square e"'] & \square(r_\alpha, e) \arrow[d, "r_\alpha \square e"] \\
X_\alpha^{\wedge n} \wedge B \arrow[r] & R_\alpha \wedge B
\end{tikzcd}$$

Let furthermore s_α be the universal morphism from the colimit R_α into the wedge-power $X_{\alpha+1}^{\wedge n}$, so that $f_{\alpha+1,0}^{\square n} = s_\alpha \circ r_\alpha$. Applying Lemma 16 to this composition and the morphism $e : A \to B$, we obtain yet another pushout square

$$\begin{tikzcd}
\square(r_\alpha, e) \arrow[r] \arrow[d, "r_\alpha \square e"'] & \square(f_{\alpha+1,0}^{\square n}, e) \arrow[d, "\varkappa_\alpha"] \\
R_\alpha \wedge B \arrow[r] & \square(s_\alpha, e)
\end{tikzcd}$$

Composing these two pushout squares we obtain that

$$f_{\alpha+1,0}^{\square n} \square e = (s_\alpha \square e) \circ \varkappa_\alpha \ .$$

This proves that W_α from Lemma 21 equals $F(\square(s_\alpha, e))$ and h_α equals $F(s_\alpha \square e)$ since F commutes with colimits.

By Lemma 15, the morphism s_α is the composition

$$f_{\alpha+1,\alpha}^{\square n} \circ \mathrm{psht}(\mathrm{cor}_{\Sigma_{n-1}\times\Sigma_1}^{\Sigma_n}(f_{\alpha+1,\alpha}^{\square(n-1)} \square f_{\alpha,0})) \circ \cdots \circ \mathrm{psht}(\mathrm{cor}_{\Sigma_1\times\Sigma_{n-1}}^{\Sigma_n}(f_{\alpha+1,\alpha} \square f_{\alpha,0}^{\square(n-1)})) \ .$$

By Lemma 17 the morphism $s_\alpha \square e$ is the composition of pushouts of the morphisms

$$\mathrm{psht}(\mathrm{cor}_{\Sigma_i\times\Sigma_{n-i}}^{\Sigma_n}(f_{\alpha+1,\alpha}^{\square i} \square f_{\alpha,0}^{\square(n-i)})) \square e \ ,$$

where $0 = 1, \ldots, n-1$. By Lemma 13,

$$\mathrm{psht}(\mathrm{cor}_{\Sigma_i\times\Sigma_{n-i}}^{\Sigma_n}(f_{\alpha+1,\alpha}^{\square i} \square f_{\alpha,0}^{\square(n-i)})) \square e = \mathrm{psht}(\mathrm{cor}_{\Sigma_i\times\Sigma_{n-i}}^{\Sigma_n}(f_{\alpha+1,\alpha}^{\square i} \square f_{\alpha,0}^{\square(n-i)}) \square e) \ .$$

Since $e = f_2^{\square n_2} \square \ldots \square f_l^{\square n_l}$, by Lemma 18 we have that

$$\begin{gathered} \mathrm{psht}(\mathrm{cor}_{\Sigma_i\times\Sigma_{n-i}}^{\Sigma_n}(f_{\alpha+1,\alpha}^{\square i} \square f_{\alpha,0}^{\square(n-i)})) \square e = \\ \mathrm{cor}_{\Sigma_i\times\Sigma_{n-i}\times\Sigma_{n_2}\times\cdots\times\Sigma_{n_l}}^{\Sigma_n\times\Sigma_{n_2}\times\cdots\times\Sigma_{n_l}}(f_{\alpha+1,\alpha}^{\square i} \square f_{\alpha,0}^{\square(n-i)} \square f_2^{\square n_2} \square \ldots \square f_l^{\square n_l}) \ . \end{gathered}$$

Since F commutes with colimits, it follows that for any ordinal α, such that $\alpha + 1 < \lambda$, the morphism h_α is a composition of pushouts of the morphisms

$$F(f_{\alpha+1,\alpha}^{\square i} \square f_{\alpha,0}^{\square(n-i)} \square f_2^{\square n_2} \square \ldots \square f_l^{\square n_l}) \ ,$$

where $i = 0, \ldots, n-1$. By the inductive hypothesis, any such morphism is a cofibration. Then h_α is a cofibration too. As we have shown above, $F(f_\infty^{\square n} \square e) = \mathrm{colim}\,(\psi)$. By Lemma 21, this morphism is a cofibration in $\mathscr{D}^{\Sigma_n\times\Sigma_{n_2}\times\cdots\times\Sigma_{n_l}}$. This finishes the proof of Theorem 7.

5. Künneth towers for cofibre sequences

Here we prove the existence of special towers of cofibrations connecting symmetric powers in cofibre sequences via the Künneth rule, provided (trivial) cofibrations are symmetrizable, Theorem 22. This suggests to introduce the concept of a categorified λ-structure in $\mathscr{C}$ and $Ho(\mathscr{C})$. Using the results from Section 3, we prove the existence of the λ-structure of left derived symmetric powers provided symmetrizability of generating (trivial) cofibrations in $\mathscr{C}$, Theorem 25 and Corollary 27. An application to categorical finite-dimensionality (with coefficients in $\mathbb{Z}$) is given in Corollary 28.

In a model category $\mathscr{D}$, if $X \to Y$ is a cofibration, then let Y/X be the colimit of the diagram $Y \leftarrow X \to *$, and if X and Y are cofibrant, then $X \to Y \to Y/X$ is a cofibre sequence in $\mathscr{D}$.

Theorem 22. *Let $\mathscr{C}$ be a closed symmetric monoidal model category, and let $X \xrightarrow{f} Y \to Z$ be a cofibre sequence in $\mathscr{C}$. Then, for any two natural numbers n and i, $i \leq n$, there is a cofibration $\square^n_{i-1}(f) \to \square^n_i(f)$ and a Σ_n-equivariant isomorphism*

$$\square^n_i/\square^n_{i-1} \simeq \mathrm{cor}^{\Sigma_n}_{\Sigma_{n-i}\times\Sigma_i}(X^{\wedge(n-i)} \wedge Z^{\wedge i})$$

in $\mathscr{C}$. If f is a symmetrizable cofibration and all symmetric powers $\mathrm{Sym}^i(X)$ are cofibrant, then the morphism $\tilde{\square}^n_{i-1}(f) \to \tilde{\square}^n_i(f)$, obtained by passing to the colimit of the action of the symmetric group Σ_n, is a cofibration, and $\tilde{\square}^n_i/\tilde{\square}^n_{i-1}$ can be computed by Künneth's rule,

$$\tilde{\square}^n_i/\tilde{\square}^n_{i-1} \simeq \mathrm{Sym}^{n-i}(X) \wedge \mathrm{Sym}^i(Z)\ .$$

If f is a symmetrizable trivial cofibration, then all the cofibrations $\tilde{\square}^n_{i-1}(f) \to \tilde{\square}^n_i(f)$ are trivial cofibrations.

Proof. The proof is similar to the proof of Lemma 15. For any $0 \leq i \leq n$ the diagram $X^{\wedge(n-i)} \wedge K^i_{i-1}(f)$ is a subdiagram in $K^n_{i-1}(f)$. Since the wedge product commutes with colimits, we obtain a universal morphism from $X^{\wedge(n-i)} \wedge \square^i_{i-1}(f)$ to $\square^n_{i-1}(f)$. Since $X^{\wedge(n-i)} \wedge Y^{\wedge i}$ is a vertex in the diagram $K^n_i(f)$, we have a morphism from $X^{\wedge(n-i)} \wedge Y^{\wedge i}$ to $\square^n_i(f)$. Finally, we have a standard morphism $X^{\wedge(n-i)} \wedge \square^i_{i-1}(f) \to X^{\wedge(n-i)} \wedge Y^{\wedge i}$. Collecting these morphisms together we get a commutative diagram

$$\begin{tikzcd}
X^{\wedge(n-i)} \wedge \square^i_{i-1}(f) \arrow[r] \arrow[d] & X^{\wedge(n-i)} \wedge Y^{\wedge i} \arrow[d] \\
\square^n_{i-1}(f) \arrow[r] & \square^n_i(f)
\end{tikzcd}$$

This is a $\Sigma_{n-i} \times \Sigma_i$-equivariant commutative diagram, which yields a Σ_n-equivariant commutative diagram

$$\begin{tikzcd}
\mathrm{cor}^{\Sigma_n}_{\Sigma_{n-i}\times\Sigma_i}(X^{\wedge(n-i)}\wedge\square^i_{i-1}(f)) \arrow[r] \arrow[d] & \mathrm{cor}^{\Sigma_n}_{\Sigma_{n-i}\times\Sigma_i}(X^{\wedge(n-i)}\wedge Y^{\wedge i}) \arrow[d] \\
\square^n_{i-1}(f) \arrow[r] & \square^n_i(f)
\end{tikzcd}$$

A straightforward verification shows that this is a pushout square.

By the pushout product axiom of a closed symmetric monoidal model category, the morphism $\square^i_{i-1}(f)\to Y^{\wedge i}$ is a cofibration and we have

$$Y^{\wedge i}/\square^i_{i-1}(f)\simeq Z^{\wedge i}\ .$$

By the same axiom, the functor $X^{\wedge(n-i)}\wedge -$ commutes with colimits and preserves cofibrations in $\mathscr{C}$ as the object X is cofibrant. Also the same is true for the functor cor, because this is a bouquet in the category $\mathscr{C}$. This implies the needed statements about $\square^n_i(f)$.

Now suppose that f is a symmetrizable (trivial) cofibration. Recall that taking a quotient over Σ_n commutes with colimits being a left adjoint functor. This gives a pushout square

$$\begin{tikzcd}
\mathrm{Sym}^{n-i}(X)\wedge\tilde{\square}^i_{i-1}(f) \arrow[r] \arrow[d] & \mathrm{Sym}^{n-i}(X)\wedge\mathrm{Sym}^i(Y) \arrow[d] \\
\tilde{\square}^n_{i-1}(f) \arrow[r] & \tilde{\square}^n_i(f)
\end{tikzcd}$$

The symmetric power $\mathrm{Sym}^{n-i}(X)$ is cofibrant by assumption. The morphism $\tilde{\square}^i_{i-1}(f)\longrightarrow \mathrm{Sym}^i(Y)$ is a (trivial) cofibration by assumption. Therefore the top morphism is a (trivial) cofibration. This finishes the proof. □

Corollary 23. *Let f be a cofibration between cofibrant objects in $\mathscr{C}$. Suppose that f is a symmetrizable cofibration, and all symmetric powers $\mathrm{Sym}^n(X)$ are cofibrant in $\mathscr{C}$. Then f is a symmetrizable trivial cofibration if and only if $\mathrm{Sym}^n(f)$ is a trivial cofibration for all $n\geq 0$.*

Proof. Consider the sequence of cofibrations

$$\mathrm{Sym}^n(X)=\tilde{\square}^n_0(f)\to\tilde{\square}^n_1(f)\to\cdots\to\tilde{\square}^n_i(f)\to\cdots\to\tilde{\square}^n_n(f)=\mathrm{Sym}^n(Y)$$

provided by Theorem 22. The composition of all the cofibrations in that chain is $\mathrm{Sym}^n(f)$. If f is a symmetrizable trivial cofibration then each cofibration

$$\tilde{\square}^n_i(f) \longrightarrow \tilde{\square}^n_{i+1}(f)$$

is a trivial cofibration by Theorem 22. Thus, so is $\mathrm{Sym}^n(f)$. Conversely, suppose $\mathrm{Sym}^n(f)$ is a trivial cofibration for any $n \geq 0$. Let's prove by induction on n that the morphism $\tilde{\square}^n_{n-1}(f) \to \mathrm{Sym}^n(Y)$ is a trivial cofibration, i.e. that f is a symmetrizable trivial cofibration. The base of induction, $n = 1$, is obvious. To make the inductive step we observe that in proving Theorem 22 we deduce that $\tilde{\square}^n_{i-1}(f) \to \tilde{\square}^n_i(f)$ is a trivial cofibration by only using that $\tilde{\square}^i_{i-1}(f) \to \mathrm{Sym}^i(Y)$ is a trivial cofibration for $i < n$. But the last condition holds by the induction hypothesis. Thus, all morphisms $\tilde{\square}^n_{i-1}(f) \to \tilde{\square}^n_i(f)$, $1 \leq i \leq n-1$, are trivial cofibrations. Then $\tilde{\square}^n_{n-1}(f) \to \mathrm{Sym}^n(Y)$ is a weak equivalence by 2-out-of-3 property for weak equivalences. Finally, by the assumption of the lemma, $\tilde{\square}^n_{n-1}(f) \to \mathrm{Sym}^n(Y)$ is a cofibration, and so a trivial cofibration. □

Definition 24. For any closed symmetric monoidal model category $\mathscr{C}$ with monoidal unit $\mathbb{1}$, a λ-*structure* on $\mathscr{C}$ is a sequence Λ of endofunctors $\Lambda^n : \mathscr{C} \to \mathscr{C}$, $n = 0, 1, 2, \ldots$, such that

(i) $\Lambda^0 = \mathbb{1}$, $\Lambda^1 = \mathrm{Id}$,
(ii) $\Lambda^n(\emptyset) = \emptyset$ for all $n \geq 1$,
(iii) to each cofibre sequence $X \to Y \to Z$ in $\mathscr{C}$ and any n there is associated a unique sequence of cofibrations between cofibrant objects

$$\Lambda^n(X) = L^n_0 \to L^n_1 \to \cdots \to L^n_i \to \cdots \to L^n_n = \Lambda^n(Y) ,$$

called *Künneth tower*, such that for each index $0 \leq i \leq n$ one has isomorphisms

$$L^n_i / L^n_{i-1} \simeq \Lambda^{n-i}(X) \wedge \Lambda^i(Z) ,$$

and
(iv) such towers are functorial in cofibre sequences in the obvious sense.

In particular, the endofunctors Λ^n preserve cofibrant objects in $\mathscr{C}$. In these terms, Theorem 22 says that if cofibrations in $\mathscr{C}$ are symmetrizable, then symmetric powers yield a specific λ-structure in $\mathscr{C}$. We will call it the canonical λ-structure of symmetric powers in $\mathscr{C}$.

A cofibre sequence in $Ho(\mathscr{C})$ is a sequence of two composable morphisms, which is isomorphic to a sequence coming from a cofibre sequence in $\mathscr{C}$ via the functor from $\mathscr{C}$ to $Ho(\mathscr{C})$. A similar definition of a λ-structure can be then given also in $Ho(\mathscr{C})$. If Λ^* is a λ-structure on $\mathscr{C}$ such that Λ^n takes trivial cofibrations between cofibrant objects into weak equivalences, then by Ken Brown's lemma the left derived functors $L\Lambda^n$ exist, and their collection gives a λ-structure in $Ho(\mathscr{C})$. Combining this with Corollary 23, we obtain the following important result.

Theorem 25. *Let $\mathscr{C}$ be a closed symmetric monoidal model category, such that all cofibrations are symmetrizable, and all trivial cofibrations between cofibrant objects are symmetrizable in $\mathscr{C}$. Then symmetric powers Sym^n take weak equivalences between cofibrant objects to weak equivalences, and the canonical λ-structure of symmetric powers in $\mathscr{C}$ induces the λ-structure of left derived symmetric powers $L\mathrm{Sym}^n$ in $Ho(\mathscr{C})$.*

Remark 26. Let $\mathscr{C}$ be a closed symmetric monoidal model category cofibrantly generated by a set of generating cofibrations I and a set of generating trivial cofibrations J. Suppose I and J are both symmetrizable. Then by Corollaries 9 and 10, the conditions of Theorem 25 are satisfied.

Assume now that $\mathscr{C}$ is moreover pointed. According to [10], there is a well-defined S^1-suspension functor $- \wedge^L S^1 : \mathscr{T} \to \mathscr{T}$ provided by a $Ho(\Delta^{op}\mathscr{S}ets_*)$-module structure on the homotopy category $\mathscr{T} = Ho(\mathscr{C})$. If it is an autoequivalence on $\mathscr{T}$ then $\mathscr{T}$ is triangulated, where the translation functor [1] is given by $- \wedge^L S^1$ and distinguished triangles come from cofibre sequences in $\mathscr{C}$, see Chapter 7 in [10]. Since $\mathscr{C}$ is closed symmetric monoidal, so is the triangulated category $\mathscr{T}$, and the functor $\mathscr{C} \to \mathscr{T}$ is monoidal as well, see Section 4.3 in [10]. We will denote the monoidal product in $\mathscr{T}$ also by $\wedge$. A λ-structure in $\mathscr{T} = Ho(\mathscr{C})$ associates Künneth towers to distinguished triangles in $Ho(\mathscr{C})$ in the functorial way. Using Theorem 25 we obtain the following result.

Corollary 27. *Let $\mathscr{T}$ be the homotopy category of a pointed closed symmetric monoidal model category $\mathscr{C}$, so that $\mathscr{T}$ is triangulated. Assume, furthermore, that all cofibrations are symmetrizable, and all trivial cofibrations between cofibrant objects are symmetrizable in $\mathscr{C}$. Then $\mathscr{T}$ inherits the canonical λ-structure of left derived symmetric powers associated to distinguished triangles in $\mathscr{T}$.*

As a straightforward consequence of Corollary 27 we also get the following corollary.

Corollary 28. *Let $\mathscr{T}$ be as above, and let $X \xrightarrow{f} Y \to Z \to X[1]$ be a distinguished triangle in $\mathscr{T}$. If there exist natural numbers n' and m' such that $L\mathrm{Sym}^n(X) = 0$ for all $n \geq n'$ and $L\mathrm{Sym}^m(Z) = 0$ for all $m \geq m'$, then there exists N' such that $L\mathrm{Sym}^N(Y) = 0$ for all $N \geq N'$.*

6. Localization of symmetric powers

In this section we prove a few results on the Bousfield localization of model categories with regard to monoidal structures and symmetric powers on them, which will be used later. In particular, Theorem 33 gives a necessary and sufficient condition for trivial cofibrations to remain symmetrizable after Bousfield localization of $\mathscr{C}$ by a set of morphisms S. This will be applied to the localization by an abstract interval in Section 7.

Let $\mathscr{C}$ be a left proper cellular model category, and denote the model structure in $\mathscr{C}$ by $\mathscr{M}$. Recall that left properness means that the pushout of a weak equivalence along a cofibration is a weak equivalence. Cellularity means that $\mathscr{C}$ is cofibrantly generated by a set of generating cofibrations I and a set of trivial generating cofibrations J, the domains and codomains of morphisms in I are all compact relative to I, the domains of morphisms in J are all small relative to the cofibrations, and cofibrations are effective monomorphisms. Further details about these notions can be found, for instance, in [10, 11] or [8].

Let S be a set of morphisms in $\mathscr{C}$. Recall that an object Z in $\mathscr{C}$ is called S-local if it is fibrant, and for any morphism $f : A \to B$ in S the morphism between function complexes

$$\mathrm{map}(f, Z) : \mathrm{map}(B, Z) \to \mathrm{map}(A, Z)$$

is a weak equivalence in $\Delta^{op}\mathscr{S}ets$, see Definition 3.1.4(1)(a) in [8]. The construction of the function complex bi-functor $\mathrm{map}(-,-)$ is given in Sections 17.1–17.4 in [8] (see also Section 5.4 in [10]). A morphism $g : X \to Y$ in $\mathscr{C}$ is said to be an S-local equivalence if the induced morphism

$$\mathrm{map}(g, Z) : \mathrm{map}(Y, Z) \to \mathrm{map}(X, Z)$$

is a weak equivalence in $\Delta^{op}\mathscr{S}ets$ for any S-local object Z in $\mathscr{C}$, see Definition 3.1.4(1)(b) in [8]. Notice that since $\mathrm{map}(-,-)$ is a homotopic invariant, each weak equivalence is an S-local equivalence in $\mathscr{C}$.

By the main result in [8] (see Theorem 4.1.1), under the assumptions above, there exists a new left proper cellular model structure $\mathscr{M}_S$ on $\mathscr{C}$ whose cofibrations remain unchanged and new weak equivalences W_S are exactly S-local equivalences in $\mathscr{C}$. The new model structure is cofibrantly generated by the set of generating cofibrations I and a new set of generating trivial cofibrations J_S, and it is called a (left) Bousfield localization of $\mathscr{M}$ with respect to S. The symbol $\mathscr{C}_S$ will be used to denote the same category $\mathscr{C}$, endowed with the new model structure $\mathscr{M}_S$. Then $\mathscr{C}_S$ is a (left) Bousfield localization of $\mathscr{C}$ with respect to S.

Let $F : \mathscr{C} \to \mathscr{D}$ be a left Quillen functor such that $F(Q(f))$ is a weak equivalence for any $f \in S$, where Q denotes the cofibrant replacement functor in the model structure $\mathscr{M}$. Then F is still left Quillen with respect to the localized model structure $\mathscr{M}_S$ and has a left derived with respect to $\mathscr{M}_S$, see Proposition 3.3.18(1) in [8]. Our main goal is to construct left derived symmetric powers for the localized model category. Since symmetric powers do not admit right adjoints in general, and thus are not left Quillen, we need to strengthen the above result.

Given a functor $F : \mathscr{C} \to \mathscr{D}$ to a model category, we say that a morphism g in $\mathscr{C}$ is F-acyclic if g is a cofibration between cofibrant objects in $\mathscr{C}$ and $F(g)$ is a weak equivalence in $\mathscr{D}$. Obviously, given composable cofibrations between cofibrant objects, their F-acyclicity has 2-out-of-3 property. By an S-local cofibration we mean a cofibration which is an S-local equivalence in $\mathscr{C}$.

Theorem 29. *Let $F : \mathscr{C} \to \mathscr{D}$ be a functor to a model category such that all trivial cofibrations between cofibrant objects in $\mathscr{M}$ are F-acyclic and $F(Q(f))$ is a weak equivalence in $\mathscr{D}$ for any $f \in S$. In addition, suppose that F-acyclic morphisms are closed under transfinite compositions and pushouts with respect to morphisms to cofibrant objects. Then all S-local cofibrations between cofibrant objects are F-acyclic. In particular, by Ken Brown's lemma, the left derived functor $LF : Ho(\mathscr{C}_S) \to Ho(\mathscr{D})$ exists and commutes with the localization functor $Ho(\mathscr{C}) \to Ho(\mathscr{C}_S)$.*

To prove Theorem 29 we first need to prove an auxiliary result. Fix a left framing on $\mathscr{C}$, see Definition 5.2.7 in [10]. Thus, for each cofibrant object X one has the functorial cofibrant replacement X^* of the constant cosimplicial object given by X, with respect to the Reedy model structure on the category of cosimplicial objects in $\mathscr{C}$. The product $X \wedge K$ in $\mathscr{C}$ of X and a simplicial set K is then defined as the product $X^* \wedge K$. For any morphism g in $\mathscr{C}$, and a morphism i in $\Delta^{op}\mathscr{S}ets$, we have their pushout product $g \square i$. For a non-negative integer m let $i_m : \partial\Delta[m] \to \Delta[m]$ be the embedding of the boundary into the m-th simplex.

Lemma 30. *Let F be as in Theorem 29. Then F-acyclic morphisms are closed under taking products with simplicial sets generated by finitely many non-degenerate simplices and pushout products with the embeddings i_m.*

Proof. Let $g : X \to Y$ be an F-acyclic morphism in $\mathscr{C}$, and let K be a simplicial set. Let m be the maximal dimension of non-degenerated simplices in K, and n be the number of such simplices. We apply induction with respect to the lexicographical order on the set of pairs (m, n). Represent K as a simplicial set obtained by gluing an m-dimensional simplex to another simplicial set K' having one simplex less than in K, i.e. $i : K' \to K$ is a pushout of i_m. By Corollary 5.4.4(1) in [10], the functor $X \wedge -$ is left Quillen. It follows that the morphism $X = X \wedge \Delta[0] \to X \wedge \Delta[m]$ is a trivial cofibration between cofibrant objects, whence it is an F-acyclic morphism by the assumption on F. Since the same is true for $Y \to Y \wedge \Delta[m]$, we see that the morphism $X \wedge \Delta[m] \to Y \wedge \Delta[m]$ is also F-acyclic by 2-out-of-3 property for acyclicity. The morphism $X \wedge \Delta[m] \to \square(g, i_m)$ is a pushout of $g \wedge \mathrm{id}_{\partial\Delta[m]}$. Then it is F-acyclic by the pushout property for acyclicity and the induction. Using 2-out-of-3 property once again, we conclude that $g \square i_m$ is F-acyclic. The obvious commutative diagram

$$\begin{tikzcd}
\square(g, i_m) \arrow[r] \arrow[d] & \square(g, i) \arrow[d] \\
Y \wedge \Delta[m] \arrow[r] & Y \wedge K
\end{tikzcd}$$

is a pushout square, and all objects in it are cofibrant. Then $g \wedge \mathrm{id}_K = \mathrm{psht}(g\square i_m) \circ \mathrm{psht}(g \wedge \mathrm{id}_{K'})$. The induction and the pushout property finish the proof of the lemma. □

Using a standard transfinite composition argument and Lemma 30 one can also show that F-acyclic morphisms are closed under products with arbitrary simplicial sets and pushout products with arbitrary cofibrations between simplicial sets, though we do not need this. Now we can prove Theorem 29.

Proof. By Ken Brown's lemma and the assumption of the theorem, F sends weak equivalences between cofibrant objects in $\mathscr{C}$ to weak equivalences in $\mathscr{D}$. For any morphism $f : A \to B$ of S decompose $Q(f)$ into a cofibration $f' : Q(A) \to C$ and a trivial fibration $f'' : C \to Q(B)$. Since f'' is a weak equivalence between cofibrant objects in $\mathscr{C}$, $F(f'')$ is a weak equivalence. Let $S' = \{f' | f \in S\}$. Then all morphisms in S' are F-acyclic. Since $\mathscr{M}_S = \mathscr{M}_{S'}$, without loss of generality, one may assume that all morphisms in S are F-acyclic.

Next, let $g : X \to Y$ be an S-local cofibration between cofibrant objects in $\mathscr{C}$. Let $L_S(g) : L_S(X) \to L_S(Y)$ be the fibrant replacement of the morphism g with respect to the localized model structure $\mathscr{M}_S$. Then $L_S(g)$ is a weak equivalence between cofibrant objects in $\mathscr{C}$, whence $F(L_S(g))$ is a weak equivalence. This gives that the theorem will be proved as soon as we prove that $X \to L_S(X)$ is F-acyclic.

By Theorem 4.3.1 in [8], the morphism $X \to L_S(X)$ is a relative Λ-cell complex, where Λ consists of morphisms that are either trivial cofibrations between cofibrant objects, or being composed with a weak equivalence between cofibrant objects are equal to $f\square(\partial\Delta[n] \to \Delta[n])$, where f runs S. By Lemma 30 and 2-out-of-3 property, all morphisms in Λ are F-acyclic and the theorem is proved by the assumptions on F. □

Now we need to investigate when the compatibility between the model and monoidal structures is stable under localization. For that we shall prove Lemma 31 below, following the ideas taken from the proofs of Theorems 6.3 and 8.11 in [11]. The same result is also proven in [25], Theorem 4.5.

Since now we assume that $\mathscr{C}$ is a closed symmetric monoidal left proper cellular model category cofibrantly generated by the set of generating cofibrations I and the set of generating trivial cofibrations J, such that the domains and codomains of the cofibrations from I are cofibrant. Let also Q be the cofibrant replacement in $\mathscr{C}$, and so in $\mathscr{C}_S$.

Lemma 31. *The model structure $\mathscr{M}_S$ is compatible with the monoidal structure in $\mathscr{C}$ if and only if for any $X \in \mathrm{dom}(I) \cup \mathrm{codom}(I)$ and for any $f \in S$ the product $X \wedge Q(f)$ is an S-local equivalence.*

Proof. If $\mathscr{M}_S$ is compatible with the monoidal structure in $\mathscr{C}$, then $X \wedge Q(f)$ is an S-local equivalence by the axioms of a monoidal model category. Conversely, let $h : X \to Y$ be a cofibration in I and let $g : Z \to U$ be an S-local cofibration in $\mathscr{C}$. By Corollary 4.2.5 in [10] all we need to show is that $h \square g$ is an S-local cofibration in $\mathscr{C}$. By Theorem 2.2 in [11], the functors $X \wedge -$ and $Y \wedge -$ are left Quillen with respect to the localized model structure $\mathscr{M}_S$. This is because $X \wedge Q(f)$ is an S-local equivalence for any f from S, and the same for $Y \wedge Q(f)$. Since $X \wedge -$ is left Quillen and $g : Z \to U$ is an S-local cofibration, the morphism $\mathrm{id}_X \wedge g : X \wedge Z \to X \wedge U$ is an S-local cofibration. Since trivial cofibrations are stable under pushouts, the pushout $Y \wedge Z \to \square(h, g)$ is an S-local cofibration too. The morphism $\mathrm{id}_Y \wedge g : Y \wedge Z \to Y \wedge U$ is an S-local cofibration, because $Y \wedge -$ is left Quillen. Since $\mathrm{id}_Y \wedge g$ is the composition $Y \wedge Z \to \square(h, g) \xrightarrow{h \square g} Y \wedge U$, we obtain that $h \square g$ is an S-local equivalence. Moreover, $h \square g$ is a cofibration since $\mathscr{C}$ monoidal model. $\square$

Remark 32. Lemma 31 has the following direct generalization. Let $\mathscr{C}$ and S be as in the lemma and $\mathscr{D}$ be a $\mathscr{C}$-module in the sense of Definition 4.2.18 in [10]. Let R be a set of morphisms in $\mathscr{D}$ and assume that $\mathscr{D}$ is left proper and cellular. Let I' be a set of generating cofibrations in $\mathscr{D}$. Suppose the condition of Lemma 31 is satisfied, for all $X \in \mathrm{dom}(I) \cup \mathrm{codom}(I)$ and $g \in R$ the product $X \wedge Q(g)$ is R-local, and for all $f \in S$ and $Y \in \mathrm{dom}(I') \cup \mathrm{codom}(I')$ the product $Q(f) \wedge Y$ is R-local as well. Then the localized model category $\mathscr{D}_R$ is a $\mathscr{C}_S$-module.

Theorem 33. *Let $\mathscr{C}$ and S be such that $\mathscr{M}_S$ is compatible with the monoidal structure in $\mathscr{C}$, and assume furthermore that all cofibrations are symmetrizable and all trivial cofibrations between cofibrant objects are symmetrizable in $\mathscr{C}$. Assume also that for any $f \in S$ and any natural n the morphism $\mathrm{Sym}^n(Q(f))$ is an S-local equivalence. Then all S-local cofibrations between cofibrant objects are symmetrizable in $\mathscr{C}_S$. The left derived functors $L\mathrm{Sym}^n$ exist on $Ho(\mathscr{C}_S)$, and they commute with the localization functor $Ho(\mathscr{C}) \to Ho(\mathscr{C}_S)$.*

Proof. Let F be the composition of Sym^n and the localization functor $\mathscr{C} \to \mathscr{C}_S$ (this is just the identity functor considered as a functor between two different model structures). Since cofibrations in $\mathscr{C}$ are symmetrizable, they are so in $\mathscr{C}_S$. By Corollary 23 applied to $\mathscr{C}_S$, we see that trivial symmetrizable cofibrations between cofibrant objects in $\mathscr{C}_S$ are the same as F-acyclic morphisms in $\mathscr{C}$. So, it is enough to show that S-local cofibrations are F-acyclic.

By Theorem 7 applied to the category $\mathscr{C}_S$, F-acyclic morphisms are closed under transfinite compositions and under pushouts with respect to morphisms to cofibrant objects (actually, to treat transfinite compositions it is enough to use Lemma 19 and Theorem 22). We conclude by Theorem 29. □

7. Localization w.r.t. diagonalizable intervals

Let us consider more closely the important particular case of the left Bousfield localization contracting an object A into a point. If A is what we call a diagonalizable interval, then, using the results from Section 6, we prove that trivial cofibrations (between cofibrant objects) remain symmetrizable in the localized category, Theorem 42. As a consequence, we obtain that left derived symmetric powers exist in the homotopy category of the localized category $\mathscr{C}_S$, provided we have them in $Ho(\mathscr{C})$, see Corollary 43. This will be applied in Section 11 to the unstable motivic homotopy theory, where A will be the affine line $\mathbb{A}^1$ over a base.

Let $\mathscr{C}$ be a closed symmetric monoidal left proper cellular model category $\mathscr{C}$ cofibrantly generated by the set of generating cofibrations I and the set of generating trivial cofibrations J, such that the domains and codomains of the cofibrations from I are cofibrant. Let A be a cofibrant object, let $\pi : A \to \mathbb{1}$ be a morphism in $\mathscr{C}$, and let

$$S = \{X \wedge A \stackrel{\mathrm{id}_X \wedge \pi}{\longrightarrow} X \mid X \in \mathrm{dom}(I) \cup \mathrm{codom}(I)\} \,.$$

For any morphism $f : X \to Y$ and any object Z in $\mathscr{C}$ the morphism $\underline{\mathrm{Hom}}(f, Z) : \underline{\mathrm{Hom}}(Y, Z) \to \underline{\mathrm{Hom}}(X, Z)$ in $\mathscr{C}$, as well as the morphism $\mathrm{map}(f, Z) : \mathrm{map}(Y, Z) \to \mathrm{map}(X, Z)$ in $\Delta^{op}\mathscr{S}ets$, will be denoted by f^*.

Notice that, if $X \in \mathrm{dom}(I) \cup \mathrm{codom}(I)$, it is cofibrant, and since A is cofibrant, the monoidal product $X \wedge A$ is cofibrant too.

The following two lemmas and Proposition 36 are well-known to experts. We give complete proofs, as we could not find them in the literature.

Lemma 34. *An object Z in $\mathscr{C}$ is S-local if and only if Z is fibrant in $\mathscr{C}$ and the induced morphism $\pi^* : Z \simeq \underline{\mathrm{Hom}}(\mathbb{1}, Z) \to \underline{\mathrm{Hom}}(A, Z)$ is a weak equivalence in $\mathscr{C}$.*

Proof. Let $X \in \mathrm{dom}(I) \cup \mathrm{codom}(I)$. If π^* is a weak equivalence, the morphism $\mathrm{map}(X, \pi^*) : \mathrm{map}(X, Z) \to \mathrm{map}(X, \underline{\mathrm{Hom}}(A, Z))$ is a weak equivalence of simplicial sets. If Z is fibrant, then the simplicial sets $\mathrm{map}(X, \underline{\mathrm{Hom}}(A, Z))$ and $\mathrm{map}(X \wedge A, Z)$ are canonically weak equivalent, since the objects X and A are cofibrant in $\mathscr{C}$. The composition of the morphism $\mathrm{map}(X, \pi^*)$ with this weak equivalence equals to the morphism $(\mathrm{id}_X \wedge \pi)^* : \mathrm{map}(X, Z) \to \mathrm{map}(X \wedge A, Z)$, so that $(\mathrm{id}_X \wedge \pi)^*$ is a weak equivalence of simplicial sets as well. By definition, it means that Z is S-local. Conversely, if Z is S-local, the morphism $(\mathrm{id}_X \wedge \pi)^*$ and so $\mathrm{map}(X, \pi^*)$ are weak equivalences of simplicial sets. Then $Z \simeq \underline{\mathrm{Hom}}(\mathbb{1}, Z) \stackrel{\pi^*}{\to} \underline{\mathrm{Hom}}(A, Z)$ is a weak equivalence in $\mathscr{C}$ by Proposition 3.2 in [11]. □

Lemma 35. *If Y is a cofibrant object in $\mathscr{C}$, the morphism $Y \wedge A \stackrel{\mathrm{id}_Y \wedge \pi}{\longrightarrow} Y \wedge \mathbb{1} \simeq Y$ is an S-local equivalence, i.e. a weak equivalence in $\mathscr{C}_S$.*

Proof. For any S-local object Z the morphism $\pi^* : Z \to \underline{\mathrm{Hom}}(A, Z)$ is a weak equivalence by Lemma 34 so that $\mathrm{map}(Y, \pi^*)$ is a weak equivalence of simplicial sets. As in the proof of Lemma 34 this implies that $(\mathrm{id}_Y \wedge \pi)^*$ is a weak equivalence of simplicial sets for any S-local Z. This means that the morphism $Y \wedge A \stackrel{\mathrm{id}_Y \wedge \pi}{\longrightarrow} Y$ is an S-local equivalence. $\square$

Proposition 36. *Let $\mathscr{C}$ and S be as above. Then the model structure $\mathscr{M}_S$ is compatible with the monoidal structure in $\mathscr{C}$.*

Proof. Let X be an object in $\mathrm{dom}(I) \cup \mathrm{codom}(I)$ and let f be a morphism from the set S. By definition, there exists $W \in \mathrm{dom}(I) \cup \mathrm{codom}(I)$, such that $f = \mathrm{id}_W \wedge \pi : W \wedge A \to W$. Smashing with X we obtain the morphism $\mathrm{id}_X \wedge f : X \wedge W \wedge A \to X \wedge W$. Applying Lemma 35 to $Y = X \wedge W$ we obtain that $\mathrm{id}_X \wedge f$ is a weak equivalence in $\mathscr{C}$. Hence, the category $\mathscr{C}$ and the set S satisfy the conditions of Lemma 31. Notice that the cofibrant replacements can be ignored here because X and W are in $\mathrm{dom}(I) \cup \mathrm{codom}(I)$, so that they are cofibrant, and A is cofibrant too. $\square$

Notice that the proof of Proposition 36 follows closely the proofs of Theorems 6.3 and 8.11 in [11].

Our aim is now to apply Theorem 33 to $\mathscr{C}_S$ with S as above. For this we need to impose more conditions on the morphism π. Suppose we are given with two morphisms $i_0, i_1 : \mathbb{1} \to A$, such that $\pi \circ i_0 = \pi \circ i_1 = \mathrm{id}_{\mathbb{1}}$. If $f, g : X \rightrightarrows Y$ are two morphisms from X to Y in $\mathscr{C}$, then we say that f and g are A-homotopic if there is a morphism $H : X \wedge A \to Y$, such that $H \circ (\mathrm{id}_X \wedge i_0) = f$ and $H \circ (\mathrm{id}_X \wedge i_1) = g$. If $f : X \to Y$ and $g : Y \to X$ are two morphisms in opposite directions, such that $g \circ f$ is A-homotopic to id_X and $f \circ g$ is A-homotopic to id_Y, then f and g are mutually inverse A-homotopy equivalences in $\mathscr{C}$.

Following [16], we will be saying that π is an *interval* if there exists a morphism $\mu : A \wedge A \to A$, such that $\mu \circ (\mathrm{id}_A \wedge i_0) = i_0 \circ \pi$ and $\mu \circ (\mathrm{id}_A \wedge i_1) = \mathrm{id}_A$ as morphisms from A to itself.

Lemma 37. *Let $\pi : A \to \mathbb{1}$ be an interval in $\mathscr{C}$. Then, for any cofibrant object X in $\mathscr{C}$, the morphism $\mathrm{id}_X \wedge \pi : X \wedge A \to X \wedge \mathbb{1} \simeq X$ is an A-homotopy equivalence in $\mathscr{C}$.*

Proof. From the definition of an interval, it follows that $(\mathrm{id}_X \wedge \pi) \circ (\mathrm{id}_X \wedge i_0) = \mathrm{id}_X$. Let $H = \mathrm{id}_X \wedge \mu$, where μ is taken from the definition of an interval for A. Then $(X \wedge A) \wedge A \simeq X \wedge (A \wedge A) \stackrel{\mathrm{id}_X \wedge \mu}{\longrightarrow} X \wedge A$ is an A-homotopy from $(\mathrm{id}_X \wedge i_0) \circ (\mathrm{id}_X \wedge \pi)$ to $\mathrm{id}_{X \wedge A}$. $\square$

Definition 38. The object A, together with the morphisms $i_0, i_1 : \mathbb{1} \to A$, is said to be *diagonalizable* if A is a symmetric co-algebra (possibly, without a co-unit), i.e. there exists a morphism $\delta : A \to A \wedge A$, such that the compositions $(\mathrm{id}_A \wedge \delta) \circ \delta$ and $(\delta \wedge \mathrm{id}_A) \circ \delta$ coincide, $\mathrm{t} \circ \delta = \delta$, where $\mathrm{t} : A \wedge A \to A \wedge A$ is the transposition in $\mathscr{C}$, and there are two equalities $\alpha \circ i_0 = (i_0 \wedge i_0) \circ \xi$ and $\alpha \circ i_1 = (i_1 \wedge i_1) \circ \xi$, where ξ is the inverse to the obvious isomorphism $\mathbb{1} \wedge \mathbb{1} \xrightarrow{\sim} \mathbb{1}$.

By co-associativity, we have also the morphisms $\delta_n : A \to A^{\wedge n}$ obtained by iterating δ. The following lemma is a straightforward generalization of Lemma 1, where $\Delta[1]$ is being replaced by a diagonalizable object A.

Lemma 39. *Let A be diagonalizable. Then, for any two A-homotopic morphisms $f, g : X \rightrightarrows Y$, and for any positive integer n, the morphisms $\mathrm{Sym}^n(f)$ and $\mathrm{Sym}^n(g)$ are A-homotopic in $\mathscr{C}$.*

Example 40. Let $\mathscr{C}$ be as above and assume furthermore that $\mathscr{C}$ is simplicial, and that the structures are compatible with each other. Consider the functor $\Delta^{op}\mathscr{S}ets \to \mathscr{C}$ sending a simplicial set K into the object $\mathbb{1} \wedge K$, and the same on morphisms. Let $\pi : A \to \mathbb{1}$ be the image of the morphism $\Delta[1] \to \Delta[0]$ under this functor. Then π is a diagonalizable interval in $\mathscr{C}$, where the morphism $\mu : \Delta[1] \times \Delta[1] \to \Delta[1]$ is induced by the multiplication $[1] \times [1] \to [1]$.

Example 41. Let B be a Noetherian separated scheme of finite Krull dimension, and let $\mathscr{C}$ be the category $\Delta^{op}Pre(\mathscr{S}m/B)$ of simplicial presheaves on the category of smooth schemes of finite type over B endowed with the stalk-wise model structure with respect to the Nisnevich or étale topology. By abuse of notation, denote by $\mathbb{A}^1$ the simplicial presheaf represented by the affine line $\mathbb{A}^1_B$ over B. The monoidal unit $\mathbb{1}$ is represented by B, as a scheme over itself. The structural morphism $\pi : \mathbb{A}^1 \to \mathbb{1}$ is then a diagonalizable interval in $\mathscr{C}$, where $\mu : \mathbb{A}^1 \wedge \mathbb{A}^1 \to \mathbb{A}^1$ is the multiplication induced by the fibre-wise multiplication in $\mathbb{A}^1_B$, see [16].

Now we are ready to prove the main result of this section.

Theorem 42. *Let $\mathscr{C}$ be a closed symmetric monoidal left proper cellular model category $\mathscr{C}$ cofibrantly generated by the set of generating cofibrations I and the set of generating trivial cofibrations J, such that the domains and codomains of the cofibrations from I are cofibrant, and the sets I and J are both symmetrizable. Let A be a cofibrant object and let $\pi : A \to \mathbb{1}$ be a diagonalizable interval in $\mathscr{C}$. Let also $S = \{X \wedge A \xrightarrow{\mathrm{id} \wedge \pi} X \mid X \in \mathrm{dom}(I) \cup \mathrm{codom}(I)\}$ be the set of morphisms in $\mathscr{C}$. Then all S-local cofibrations between cofibrant objects are symmetrizable.*

Proof. By Proposition 36 and Theorem 7, $\mathscr{C}$ and S satisfy the first two assumptions of Theorem 33, so that we only need to show that they satisfy the third assumption of it. By Theorem 25, symmetric powers preserve weak equivalences between cofibrant objects in $\mathscr{C}$. This is why, for any $f \in S$, the morphism $\mathrm{Sym}^n(Q(f))$ is an S-local equivalence if and only if the morphism $\mathrm{Sym}^n(f)$ is an S-local equivalence in $\mathscr{C}$.

Let now f be the morphism $\mathrm{id}_X \wedge \pi : X \wedge A \to X \wedge \mathbb{1} \simeq X$ in S, where $X \in \mathrm{dom}(I) \cup \mathrm{codom}(I)$. Then $f = \mathrm{id}_X \wedge \pi$ is an A-homotopy equivalence by Lemma 37. By Lemma 39, $\mathrm{Sym}^n(f)$ is an A-homotopy equivalence too. Since I is symmetrizable, $\mathrm{Sym}^n(X \wedge A)$ and $\mathrm{Sym}^n(X)$ are cofibrant by Corollary 10, because X and A are cofibrant.

By Proposition 36, $\mathrm{id}_Y \wedge \pi$ is an S-local equivalence for any cofibrant Y. This implies that A-homotopic morphisms between cofibrant objects are the same in the homotopy category $Ho(\mathscr{C}_S)$. Therefore, an A-homotopy between cobibrant objects is an S-local equivalence in $\mathscr{C}$. Summing up, we obtain that $\mathrm{Sym}^n(f)$ is an S-local equivalence in $\mathscr{C}$. $\square$

Corollary 43. *If the assumptions of Theorem 42 are satisfied, the left derived functors $L\mathrm{Sym}^n$ exist on $Ho(\mathscr{C}_S)$ and commute with $Ho(\mathscr{C}) \to Ho(\mathscr{C}_S)$.*

Proof. Follows from Theorem 42 and Theorem 25. $\square$

8. Positive model structures on spectra

Now we are going to study symmetric powers in stable categories. In this section we give an outline of the utmost generalization of topological and motivic positive model structures developed, respectively, in [3] and [9]. More details on abstract positive model structures can be found in [4]. Positive model structures will play the key role in Section 9.

Let $\mathscr{C}$ be a closed symmetric monoidal model category which is, moreover, left proper and cellular model category. Suppose in addition that all domains of the generating cofibrations in I are cofibrant. Let T be a cofibrant object in $\mathscr{C}$. As it was shown in [11], with the above collection of structures imposed upon $\mathscr{C}$ there is a passage from $\mathscr{C}$ to a category

$$\mathscr{S} = \mathrm{Spt}^{\Sigma}(\mathscr{C}, T)$$

of symmetric spectra over $\mathscr{C}$ stabilizing the functor

$$- \wedge T : \mathscr{C} \longrightarrow \mathscr{C} \ .$$

Let's remind the basics of this construction for reader's sake. Let Σ be a disjoint union of symmetric groups Σ_n for all $n \geq 0$, where Σ_0 is the permutation of the empty set, so, isomorphic to Σ_1, and all groups are considered as one object categories. Let $\mathscr{C}^{\Sigma}$ be the category of symmetric sequences over $\mathscr{C}$, i.e. functors from Σ to $\mathscr{C}$. Explicitly, a symmetric sequence is a collection $(X_0, X_1, X_2, \dots)$ of objects in $\mathscr{C}$ together with the action of Σ_n on X_n for each $n \geqslant 1$. Since $\mathscr{C}$ is closed symmetric monoidal, so is the category $\mathscr{C}^{\Sigma}$ with the monoidal product given by the formula

$$(X \wedge Y)_n = \vee_{i+j=n} \Sigma_n \times_{\Sigma_i \times \Sigma_j} (X_i \wedge Y_j) \ ,$$

where for any group G and a subgroup H in G the functor $G \times_H -$ is the functor cor_H^G described in Section 4, see [12] or [11]. The restriction to the n-th slice of the symmetry isomorphism $X \wedge Y \simeq Y \wedge X$ is equal to the product of the right translation

$$\Sigma_n \to \Sigma_n \;, \quad \sigma \mapsto \sigma \circ \tau_{j,i} \;,$$

and the symmetry isomorphism $X_i \wedge Y_j \simeq Y_j \wedge X_i$ in $\mathscr{C}$, where $\tau_{j,i}$ permutes the first block of j and the second block of i elements, [12, Sect. 2.1].

Let $S(T)$ be the free commutative monoid on the symmetric sequence $(\emptyset, T, \emptyset, \emptyset, \dots)$, i.e. the symmetric sequence $S(T) = (\mathbb{1}, T, T^{\wedge 2}, T^{\wedge 3}, \dots)$, where Σ_n acts on $T^{\wedge n}$ by permutation of factors (recall that $\emptyset$ is the initial object in $\mathscr{C}$). Then $\mathscr{S}$ is the category of modules over $S(T)$ in $\mathscr{C}^\Sigma$. In particular, any symmetric spectrum X is a sequence of objects $(X_0, X_1, X_2, \dots)$ in $\mathscr{C}$ together with Σ_n-equivariant morphisms

$$X_n \wedge T \longrightarrow X_{n+1} \;,$$

such that for all $n, i \geq 0$ the composite

$$X_n \wedge T^{\wedge i} \longrightarrow X_{n+1} \wedge T^{\wedge (i-1)} \to \dots \to X_{n+i}$$

is $\Sigma_n \times \Sigma_i$-equivariant. One has a natural closed symmetric monoidal structure on $\mathscr{S}$ given by product of modules over the commutative monoid $S(T)$.

For any non-negative n consider the evaluation functor

$$\mathrm{Ev}_n : \mathscr{S} \longrightarrow \mathscr{C}$$

sending any symmetric spectrum X to its n-slice X_n. Each Ev_n has a left adjoint

$$F_n : \mathscr{C} \longrightarrow \mathscr{S} \;,$$

which can be constructed as follows. First we define a naive functor $\tilde{F}_n$ from $\mathscr{C}$ to $\mathscr{C}^\Sigma$ taking any object A in $\mathscr{C}$ into the symmetric sequence

$$(\emptyset, \dots, \emptyset, \Sigma_n \times A, \emptyset, \emptyset, \dots) \;,$$

in which $\Sigma_n \times A$ stays on the n-th place. On the second stage we set

$$F_n(A) = \tilde{F}_n(A) \wedge S(T) \;,$$

see [11, Def. 7.3]. Then, for any non-negative integer m one has

$$\mathrm{Ev}_m(F_n(A)) = \Sigma_m \times_{\Sigma_{m-n}} (A \wedge T^{\wedge (m-n)}) \;,$$

where Σ_{m-n} is embedded into Σ_m by permuting the first $m-n$ elements in the set $\{1, \dots, m\}$.

The functors F_n have the following monoidal property: there is a canonical isomorphism $F_p(A) \wedge F_q(B) \simeq F_{p+q}(A \wedge B)$. The restriction to the m-th slice of the symmetry isomorphism $F_p(A) \wedge F_q(B) \simeq F_q(B) \wedge F_p(A)$ is the morphism

$$\Sigma_m \times_{\Sigma_{m-p-q}} (A \wedge B \wedge T^{\wedge(m-p-q)}) \to \Sigma_m \times_{\Sigma_{m-p-q}} (B \wedge A \wedge T^{\wedge(m-p-q)})$$

which is equal to the product of the right translation

$$\Sigma_m \to \Sigma_m \ , \quad \sigma \mapsto \sigma \circ \tau_{q,p} \ ,$$

the symmetry isomorphism $A \wedge B \simeq B \wedge A$ in $\mathscr{C}$, and the identity morphism on $T^{\wedge(m-p-q)}$.

The model structure on $\mathscr{S}$ is constructed in two steps – projective model structure coming from the model structure on $\mathscr{C}$ and its subsequent Bousfield localization.

Let $I_T = \cup_{n\geq 0} F_n(I)$ and $J_T = \cup_{n\geq 0} F_n(J)$, where $F_n(I)$ is the set of all morphisms of type $F_n(f)$, $f \in I$, and the same for $F_n(J)$. Let also W_T be the set of projective weak equivalences, where a morphism $f : X \to Y$ is a projective weak equivalence in $\mathscr{S}$ if and only if $f_n : X_n \to Y_n$ is a weak equivalence in $\mathscr{C}$ for all $n \geq 0$. The projective model structure

$$\mathscr{M} = (I_T, J_T, W_T)$$

is generated by the set of generating cofibrations I_T and the set of generating weak cofibrations J_T. As the model structure in $\mathscr{C}$ is left proper and cellular, the projective model structure in $\mathscr{S}$ is left proper and cellular too, [11]. Projective fibrations of spectra are level-wise fibrations. The closed monoidal structure on $\mathscr{S}$ is compatible with the model structure $\mathscr{M}$.

Remark 44. By Remark 7.4. in [11], each functor Ev_m has right adjoint. The above formula for $\mathrm{Ev}_m(F_n(A))$ implies that, given a morphism f in $\mathscr{C}$, the morphism $\mathrm{Ev}_m(F_n(f))$ is a coproduct of the product of f with a power of T. Since T is cofibrant, $\mathrm{Ev}_m(F_n(f))$ is a (trivial) cofibration provided f is so. This is why Ev_m sends generating (trivial) cofibrations, in the sense of the model structure $\mathscr{M}$, to (trivial) cofibrations in the model category $\mathscr{C}$. Applying Lemma 2.1.20 in [10], we see that the functors Ev_m are left Quillen.

Let now

$$\zeta_n^A : F_{n+1}(A \wedge T) \to F_n(A)$$

be the adjoint to the morphism

$$A \wedge T \to \mathrm{Ev}_{n+1}(F_n(A)) = \Sigma_{n+1} \times (A \wedge T)$$

induced by the canonical embedding of Σ_1 into Σ_{n+1}. For any set of morphisms U let $\mathrm{dom}(U)$ and $\mathrm{codom}(I)$ be the set of domains and codomains of morphisms from U, respectively. Let then

$$S = \{\zeta_n^A \mid A \in \mathrm{dom}(I) \cup \mathrm{codom}(I)\ ,\ \ n \geq 0\}\ ,$$

where Q is the cofibrant replacement in the projective model structure. Then a stable model structure

$$\mathscr{M}_S = (I_T, J_{T,S}, W_{T,S})$$

in $\mathscr{S}$ is defined to be the Bousfield localization of the projective model structure with respect to the set S. It is generated by the same set of generating cofibrations I_T, and by a new set of generating weak cofibrations $J_{T,S}$. Here $W_{T,S}$ is the set of stable weak equivalences, i.e. new weak equivalences obtained as a result of the localization. The condition of Lemma 31 is satisfied and the stable model structure is compatible with the monoidal structure on $\mathscr{S}$.

The importance of the stable model structure is that the functor $- \wedge T$ is a Quillen autoequivalence of $\mathscr{S}$ with respect to this model structure.

An abstract stable homotopy category, in our understanding, is the homotopy category $\mathscr{T}$ of the category of symmetric spectra over a closed symmetric monoidal model category $\mathscr{C}$ as above, stabilizing a smash-with-T functor for a cofibrant object T in $\mathscr{C}$, i.e. the homotopy category of $\mathscr{S}$ with respect to stable weak equivalences $W_{T,S}$.

Notice also that by Hovey's result, see [11], the homotopy category $\mathscr{T}$ is equivalent to the homotopy category of ordinary T-spectra provided the cyclic permutation on $T \wedge T \wedge T$ is left homotopic to the identity morphism.

Now we introduce positive model structures on $\mathscr{S}$. Let $I_T^+ = \cup_{n>0} F_n(I)$, $J_T^+ = \cup_{n>0} F_n(J)$ and let W_T^+ be the set of morphisms $f : X \to Y$, such that $f_n : X_n \to Y_n$ is a weak equivalence in $\mathscr{C}$ for all $n > 0$. We call such morphisms positive projective weak equivalences.

Proposition 45. *There is a cofibrantly generated model structure on $\mathscr{S}$*

$$\mathscr{M}^+ = (I_T^+, J_T^+, W_T^+)\ ,$$

called a positive projective model structure. Positive projective fibrations are level-wise fibrations in positive levels. Positive projective cofibrations are projective cofibrations that are also isomorphisms in the zero level.

Proof. We check that the sets I_T^+, J_T^+ and W_T^+ satisfy the conditions of Theorem 2.1.19 in [10], so that they generate a model structure. Condition 1 is satisfied automatically. Conditions 2 and 3 are immediately implied by the inclusions I_T^+-cell $\subset I_T$-cell, J_T^+-cell $\subset J_T$-cell and the fact that $\mathscr{M} = (I_T, J_T, W_T)$, whence the sets I_T, J_T and W_T satisfy the conditions 2 and 3.

Obviously, all morphisms in J_T^+-cell are positive level weak equivalences. To check condition 4 it remains only to show that J_T^+-cell $\subset I_T^+$-cof. The class I_T^+-cof is closed under transfinite compositions and pushouts, see the proof of Lemma 2.1.10 on page 31 in [10].

Thus, it is enough to show that $J_T^+ \subset I_T^+$-cof, or, equivalently, that I_T^+-inj $\subset J_T^+$-inj. Since the functors (F_n, Ev_n) are adjoint, we get that

$$J_T^+\text{-inj} = \{f : X \to Y \text{ in } \mathscr{S} \mid \forall n > 0 \ \mathrm{Ev}_n(f) \text{ is a fibration in } \mathscr{C}\} ,$$

i.e. the class J_T^+-inj is the class of positive level fibrations in $\mathscr{S}$. Similarly,

$$I_T^+\text{-inj} = \{f : X \to Y \text{ in } \mathscr{S} \mid \forall n > 0 \ \mathrm{Ev}_n(f) \text{ is a trivial fibration in } \mathscr{C}\} .$$

It follows that I_T^+-inj $\subset J_T^+$-inj and condition 4 is done. Also, we obtain that J_T^+-inj $\cap$ $W_T^+ = I_T^+$-inj, which gives conditions 5 and 6.

The structure of fibrations and cofibrations in $\mathscr{M}^+$ can be proved using the definition of I_T^+, J_T^+, left lifting property and the adjunction between F_n and Ev_n. $\square$

Corollary 46. *There is a Quillen adjunction*

$$(F_1(T) \wedge -, \underline{\mathrm{Hom}}(F_1(T), -))$$

between $\mathscr{M}$ and $\mathscr{M}^+$ and a Quillen adjunction $(\mathrm{Id}, \mathrm{Id})$ *between $\mathscr{M}^+$ and $\mathscr{M}$.*

Let now

$$S^+ = \{\zeta_n^A \mid A \in \mathrm{dom}(I) \cup \mathrm{codom}(I) , \quad n > 0\} .$$

Definition 47. The localization

$$\mathscr{M}_{S^+}^+ = (I_T^+, J_{T,S^+}^+, W_{T,S^+}^+)$$

of the positive projective model structure with respect to the set S^+ will be called a *positive stable model structure* on $\mathscr{S}$.

Certainly, we can also localize the positive projective model structure by the set S getting an intermediate model structure $\mathscr{M}_S^+ = (I_T^+, J_{T,S}^+, W_{T,S}^+)$.

Lemma 48. *With respect to the closed monoidal structure on $\mathscr{S}$ the model structure $\mathscr{M}^+$ is an $\mathscr{M}$-module and the model structure $\mathscr{M}_{S^+}^+$ is an $\mathscr{M}_S$-module. In addition, the closed monoidal structure on $\mathscr{S}$ defines an adjunction in two variables with respect to both model structures $\mathscr{M}^+$ and $\mathscr{M}_{S^+}^+$ (see Definition 4.2.1 in [10]).*

Proof. The proof of the facts that $\mathscr{M}^+$ is an $\mathscr{M}$-module and that we have an adjunction in two variables with respect to $\mathscr{M}^+$ is similar to the proof of Theorem 8.3 in [11]. Then we use Lemma 31 and Remark 32. Namely, the domains and codomains of morphisms in I_T are of the form $F_n(A)$, $n \geq 0$, where A is a domain or a codomain of a morphism in I. Morphisms in S have cofibrant domains and codomains. The analogous is true in the positive setup. Now everything follows from the monoidal properties of the functors F_n. $\square$

Notice that the unit axiom is not satisfied for the model structure $\mathscr{M}^+$, thus $\mathscr{S}$ is not a closed monoidal model category with respect to $\mathscr{M}^+$. Indeed, let $S(T)^+$ denote the spectrum with $S(T)^+_0 = \emptyset$ and $S(T)^+_n = S(T)_n$ for $n > 0$. Then the natural morphism $S(T)^+ \to S(T)$ is a positive cofibrant replacement for the unit in $\mathscr{S}$. However, in general $S(T)^+ \wedge X \to X$ is not a positive weak equivalence for a positively cofibrant X. For example, if $X = F_n(A)$, $n > 0$, then a calculation shows that $(S(T)^+ \wedge F_n(A))_m = \emptyset$ for $m \leq n$ and $(S(T)^+ \wedge F_n(A))_m = (S(T) \wedge F_n(A))_m$ for $m > n$. Thus, the morphism in question fails to be a weak equivalence in level n.

Lemma 49. *Any positive weak equivalence is a stable weak equivalence.*

Proof. Let $f : X \to Y$ be a positive weak equivalence. We claim that for any Z in $\mathscr{S}$, there is a canonical bijection

$$\mathrm{Hom}_{Ho(\mathscr{M})}(Z \wedge^L F_1(T), X) = \mathrm{Hom}_{Ho(\mathscr{M})}(Z \wedge^L F_1(T), Y) .$$

For this we use Quillen adjunctions from Corollary 46 and the fact that $R\underline{\mathrm{Hom}}(F_1(T), f)$ is an isomorphism in $Ho(\mathscr{M})$ as f is an isomorphism in $Ho(\mathscr{M}^+)$.

Let $g : Y \wedge^L F_1(T) \to X$ be a morphism in $Ho(\mathscr{M})$ that corresponds to the morphism $\mathrm{id}_Y \wedge^L \zeta_0^{\mathbb{1}} : Y \wedge^L F_1(T) \to Y$ under the above bijection applied to $Z = Y$ (note that g may be not a class of a morphisms in $\mathscr{C}$, which is the reason to consider homotopy categories). Then we obtain a commutative diagram

$$\begin{tikzcd}
X \wedge^L F_1(T) \arrow[rr, "f \wedge^L \mathrm{id}"] \arrow[d, "\mathrm{id}_X \wedge^L \zeta_0^{\mathbb{1}}"'] & & Y \wedge^L F_1(T) \arrow[lld, "g"'] \arrow[d, "\mathrm{id}_Y \wedge^L \zeta_0^{\mathbb{1}}"] \\
X \arrow[rr, "f"] & & Y
\end{tikzcd}$$

The commutativity of the lower triangle is by construction of g, while commutativity of the upper triangle is checked by applying f and using the above bijection for the case $Z = X$. Since $\mathrm{id} \wedge^L \zeta_0^{\mathbb{1}}$ is an isomorphism in $Ho(\mathscr{M}_S)$, we see that f is also an isomorphism in $Ho(\mathscr{M}_S)$ with the inverse being $g \circ (\mathrm{id}_Y \wedge^L \zeta_0^{\mathbb{1}})^{-1}$. $\square$

Theorem 50. *In the above terms,*

$$W_{T,S} = W^+_{T,S^+} = W^+_{T,S} \, .$$

Proof. Let's apply Theorem 3.3.20(1)(a) from [8] to adjunctions from Corollary 46. Indeed, the domains and codomains of morphisms in S and S^+ are cofibrant in the corresponding model structures and we have $F_1(T) \wedge S \subset S^+$, $S^+ \subset S$, whence the conditions of the above theorem are satisfied. Therefore, we obtain the corresponding Quillen adjunctions between Bousfield localizations $\mathscr{M}_S$ and $\mathscr{M}^+_{S^+}$.

We claim that these localized Quillen adjunctions are actually equivalences. More precisely, the functors

$$F_1(T) \wedge^L - : Ho(\mathscr{M}_S) \to Ho(\mathscr{M}^+_{S^+}) \,, \quad L\mathrm{Id} : Ho(\mathscr{M}^+_{S^+}) \to Ho(\mathscr{M}_S)$$

are quasiinverse. For this it is enough to show that for any (positively) cofibrant X the natural morphism $F_1(T) \wedge X \to X$ is a (positive) stable weak equivalence. This follows from Lemma 48, because $F_1(T) \to F_0(\mathbb{1})$ is a stable weak equivalence.

Since cofibrant objects in $\mathscr{M}^+_{S^+}$ are the same as in $\mathscr{M}^+$, the equivalence $L\mathrm{Id}$: $Ho(\mathscr{M}^+_{S^+}) \to Ho(\mathscr{M}_S)$ sends an object X in $\mathscr{S}$ to $Q^+(X)$, where Q^+ is the cofibrant replacement in $\mathscr{M}^+$. Therefore a morphism $f : X \to Y$ in $\mathscr{S}$ is in W^+_{T,S^+} if and only if $Q^+(f)$ is in $W_{T,S}$. By Lemma 49, the natural morphisms $Q^+(X) \to X$ and $Q^+(Y) \to Y$ are in $W_{T,S}$. Consequently, $Q^+(f)$ is in $W_{T,S}$ if and only if f is in $W_{T,S}$, whence we get $W^+_{T,S^+} = W_{T,S}$. This implies that $(\mathscr{M}^+_{S^+})_S = \mathscr{M}^+_{S^+}$. On the other hand, $(\mathscr{M}^+_{S^+})_S = \mathscr{M}^+_S$, because $S^+ \subset S$. □

Corollary 51. *The monoidal structure on $\mathscr{S}$ is compatible with the model structure $\mathscr{M}^+_{S^+}$.*

Proof. By Theorem 50, the morphism $F_1(T) \to F_0(\mathbb{1})$ is a cofibrant replacement in $\mathscr{M}^+_{S^+}$. The morphism $F_1(T) \wedge X \to F_0(\mathbb{1}) \wedge X = X$ is a positive stable weak equivalence for any positively cofibrant X by Lemma 48. □

Remark 52. For a natural p call a p-level weak equivalence (fibration) a morphism in $\mathscr{S}$ which is a level weak equivalence (fibration) for n-slices with $n \geq p$. These two classes of morphisms define a model structure $\mathscr{M}^{\geq p}$ on $\mathscr{S}$. Cofibrations in $\mathscr{M}^{\geq p}$ are cofibrations in $\mathscr{M}$ which are isomorphisms on n-slices with $n < p$. By methods similar to those used above one shows that any n-level weak equivalence is a stable weak equivalence. Moreover, stable weak equivalences are obtained by localization of $\mathscr{M}^{\geq p}$ over the set of morphisms $\{\zeta^A_n \mid A \in \mathrm{dom}(I) \cup \mathrm{codom}(I) \,,\ n \geq p\}$.

9. Symmetric powers in stable categories

Using results from Section 8, we are now going to show that left derived powers exist and coincide with homotopy symmetric powers for abstract symmetric spectra, see Theorem 55 below. This will be applied in Section 11 to the motivic stable homotopy category of schemes over a base.

So, let again $\mathscr{C}$ be a closed symmetric monoidal left proper cellular model category, T a cofibrant object in $\mathscr{C}$, and $\mathscr{S} = \mathrm{Spt}^{\Sigma}(\mathscr{C}, T)$ the category of symmetric spectra.

To obtain results for symmetric spectra, similar to Theorem 25 and Corollary 27, we would require symmetrizability of generating cofibrations in $\mathscr{S}$. However, we can unlikely meet such symmetrizability in applications, see Remark 58 below. Instead, we will be exploring strong Ev_n-symmetrizability for cofibrations in $\mathscr{S}$. The phenomenon of strong Ev_n-symmetrizability was first observed in [3] for topological spectra. However, our proof for the case of abstract spectra is different from the one in [3], and heavily relies on Theorem 7.

Proposition 53. *Let X be an object in $\mathscr{S} = \mathrm{Spt}^{\Sigma}(\mathscr{C}, T)$, cofibrant with respect to the positive projective model structure $\mathscr{M}^+$. Then, for any two positive integers m and n, the object $(X^{\wedge n})_m$, as an object of the category $\mathscr{C}^{\Sigma_n}$, is cofibrant in the canonical model structure in $\mathscr{C}^{\Sigma_n}$.*

Proof. By Corollary 11, we need only to show that I_T^+ is a strongly symmetrizable set of Ev_m-cofibrations for all $m > 0$. Let $f_1, \dots, f_l$ be a finite collection of morphisms in I. Recall that I is the set of generating cofibrations in the initial cofibrantly generated category $\mathscr{C}$. Let also $p_1, \dots, p_l$ be a collection of l positive integers. We have to show that the morphism

$$\mathrm{Ev}_m((F_{p_1} f_1)^{\square n_1} \square \dots \square (F_{p_l} f_l)^{\square n_l})$$

is a cofibration in $\mathscr{C}^{\Sigma_{n_1} \times \dots \times \Sigma_{n_l}}$ for any multidegree $\{n_1, \dots, n_l\}$.

Let $r = n_1 p_1 + \dots + n_l p_l$, $f = f_1^{\square n_1} \square \dots \square f_l^{\square n_l}$, and let A and B be the source and target of the morphism f. For any non-negative i the functor F_i commutes with colimits since it is left adjoint. This and the monoidal properties of the functors F_i imply that

$$(F_{p_1} f_1)^{\square n_1} \square \dots \square (F_{p_l} f_l)^{\square n_l} = F_{n_1 p_1 + \dots + n_l p_l}(f_1^{\square n_1} \square \dots \square f_l^{\square n_l}) = F_r(f) \; .$$

Applying Ev_m one has

$$\mathrm{Ev}_m(F_r(A)) = \Sigma_m \times_{\Sigma_{m-r}} (A \wedge T^{\wedge(m-r)})$$

and

$$\mathrm{Ev}_m(F_r(B)) = \Sigma_m \times_{\Sigma_{m-r}} (B \wedge T^{\wedge(m-r)}) \; ,$$

where the group $\Sigma_{n_1} \times \dots \times \Sigma_{n_l}$ acts on A and B naturally, acts identically on $T^{\wedge(m-r)}$, and it acts by right translations on Σ_m being embedded in it as permutations of the blocks in each of the l clusters of blocks, such that the i-th cluster contains n_i blocks of p_i elements each one, for $i = 1, \dots, l$.

The point here is that this action of the group $\Sigma_{n_1} \times \dots \times \Sigma_{n_l}$ on the set $\{1, \dots, m\}$ induces a free action of the same group on the objects $\mathrm{Ev}_m(F_r(A))$ and $\mathrm{Ev}_m(F_r(B))$

because the (right) action of $\Sigma_{n_1} \times \cdots \times \Sigma_{n_l}$ on the right cosets of Σ_{m-r} in Σ_m is free.[1] It follows that the morphism $\mathrm{Ev}_m(F_r(f))$ in $\mathscr{C}^{\Sigma_{n_1} \times \cdots \times \Sigma_{n_l}}$ is isomorphic to a bouquet of several copies of the morphism $(\Sigma_{n_1} \times \cdots \times \Sigma_{n_l}) \times (f \wedge T^{\wedge(m-r)})$. Therefore, $\mathrm{Ev}_m(F_r(f))$ is a cofibration in $\mathscr{C}^{\Sigma_{n_1} \times \cdots \times \Sigma_{n_l}}$, as required. □

Let now $\mathscr{D}$ be a cofibrantly generated model category and let G be a finite group. Then the functor $Y \mapsto Y/G$ from $\mathscr{D}^G$ to $\mathscr{D}$ is left Quillen and it has left derived by Theorem 11.6.8 in [8]. Given Y in $\mathscr{D}^G$, the homotopy quotient $(Y/G)_h$ is the value of this left derived functor at Y. In particular, there is a canonical morphism from $(Y/G)_h$ to Y/G, which is a weak equivalence when Y is cofibrant in $\mathscr{D}^G$. If $\mathscr{D}$ is in addition simplicial, then the homotopy quotient $(Y/G)_h$ is weak equivalent to the Borel construction $(EG \wedge Y)/G$.

Lemma 54. *Let Y be an object in $\mathscr{S}^G$, such that for any positive integer m the object Y_m is cofibrant in the model structure on $\mathscr{C}^G$. Then the canonical morphism $(Y/G)_h \to Y/G$ is a weak equivalence in $\mathscr{M}^+$.*

Proof. Consider the positive projective model structure $\mathscr{M}^+$ on the category $\mathscr{S}$ and the induced model structure on $\mathscr{S}^G$. Let $Q_+^G(Y) \to Y$ be the cofibrant replacement in $\mathscr{S}^G$. By Remark 44 and Proposition 45, the functors Ev_m are left Quillen. Lemma 11.6.4 in [8] implies that the functors $\mathrm{Ev}_m^G : \mathscr{S}^G \to \mathscr{C}^G$ are also left Quillen. Therefore, the object $\mathrm{Ev}_m(Q_+^G(Y)) = Q_+^G(Y)_m$ is cofibrant in $\mathscr{C}^G$ for all m. Combining this with the assumption of the lemma, we see that, for all $m > 0$, the canonical morphism $Q_+^G(Y)_m/G \to Y_m/G$ is a weak equivalence in $\mathscr{C}$. As colimits in spectra are term-wise, the canonical morphism $Q_+^G(Y)/G \to Y/G$ is a positive projective weak equivalence. □

Notice that Lemma 54 is also true for the usual projective model structure $\mathscr{M}$, and for more general model structures $\mathscr{M}^{\geq p}$ from Remark 52.

Let now $\mathrm{Sym}^n(X)_h$ be the n-th homotopy symmetric power of X, i.e. the homotopy quotient $(X^{\wedge n}/\Sigma_n)_h$. Combining Proposition 53 and Lemma 54, we obtain the following important result.

Theorem 55. *Let X be an object in $\mathscr{S} = \mathrm{Spt}^\Sigma(\mathscr{C}, T)$, cofibrant with respect to the positive projective model structure $\mathscr{M}^+$. Then, for any non-negative integer n the natural morphism*

$$\theta_{X,n} : \mathrm{Sym}_h^n(X) \to \mathrm{Sym}^n(X)$$

is a weak equivalence in $\mathscr{M}^+$. Hence, it is also a stable weak equivalence by Theorem 50.

[1] It is essential that all p_i are positive.

Corollary 56. *Symmetric powers preserve positive projective and stable weak equivalences between positively cofibrant objects in $\mathscr{S}$.*

Proof. The functors Sym_h^n, being homotopy quotients, preserve positive projective and stable weak equivalences. Then we apply Theorem 55. $\square$

Corollary 57. *Let $\mathscr{T}$ be the homotopy category of the category of symmetric spectra $\mathscr{S}$. The functors $\mathrm{Sym}^n : \mathscr{S} \to \mathscr{S}$ have left derived functors $L\mathrm{Sym}^n : \mathscr{T} \to \mathscr{T}$, which are canonically isomorphic to the homotopy symmetric powers Sym_h^n. Besides, the left derived functors $L\mathrm{Sym}^n$ give a λ-structure in $\mathscr{T}$, which is canonical in the sense of positive stable model structure on symmetric spectra.*

Proof. This is a straightforward consequence of Theorem 55, Ken Brown's lemma and the fact that homotopy symmetric powers give rise to Künneth towers in distinguished triangles. $\square$

Remark 58. In contrast to level-wise strong symmetrizability asserted by Proposition 53, (positive) cofibrations in $\mathscr{S}$ are not symmetrizable in general. Indeed, if f is a cofibration in $\mathscr{C}$, then symmetrizability of $F_p(f)$ in $\mathscr{S}$, for some $p > 0$, is equivalent to strong symmetrizability of f in $\mathscr{C}$. Then cofibrations are not symmetrizable for spectra of simiplicial sets by Example 5. Furthermore, by a similar argument as in Corollaries 56 and 57, one shows that strong symmetrizability of cofibrations in $\mathscr{C}$ implies that left derived symmetric powers exist for $\mathscr{C}$ and coincide with the corresponding homotopy symmetric powers. By results from Sections 10 and 11, this gives again that cofibrations are not strongly symmetrizable for (pointed) simplicial sets and, as a consequence, for (pointed) motivic spaces (motivic spaces will be considered in Section 11 below).

10. Symmetrizable cofibrations in topology

Let us illustrate symmetrizability of (trivial) cofibrations in Kelley spaces and simiplicial sets. Recall that the category $\mathscr{T}op$ of all topological spaces is not a closed symmetric monoidal category, as it does not have an internal Hom in it. The right category is the category of Kelley spaces $\mathscr{K}$, see Definition 2.4.21(3) in [10]. It is a closed symmetric monoidal model category with regard to the monoidal product defined by means of the right adjoint to the embedding of $\mathscr{K}$ into $\mathscr{T}op$, see Theorem 2.4.23 and Proposition 4.2.11 in [10]. The point here is that the realization functor $|\ |$ from $\Delta^{op}\mathscr{S}ets$ to $\mathscr{T}op$ takes its values in $\mathscr{K}$ and, moreover, the it is symmetric monoidal left Quillen, as a functor into $\mathscr{K}$, see Proposition 4.2.17 in [10]. It follows that the category $\mathscr{K}$ is simplicial. For any non-negative integer n let $\Delta[n] = \mathrm{Hom}_\Delta(-, [n])$ be the n-th simplex. If I_s is the set of the canonical inclusions $\partial\Delta[n] \hookrightarrow \Delta[n]$, $n \geq 0$, and J_s is the set of canonical inclusions $\Lambda_i[n] \hookrightarrow \Delta[n]$, $n > 0$, $0 \leq i \leq n$, then I_s and J_s are the sets of generating cofibrations and the sets of generating trivial cofibrations for the model structure in $\Delta^{op}\mathscr{S}ets$. The sets $|I_s| = I$ and $|J_s| = J$ cofibrantly generate $\mathscr{K}$.

Lemma 59. *If f is a weak equivalence in $\Delta^{op}\mathscr{S}ets$ then* $\mathrm{Sym}^n(f)$ *is a weak equivalence in $\Delta^{op}\mathscr{S}ets$ for any $n \geq 0$.*

Proof. Let $f : X \to Y$ be a weak equivalence in $\Delta^{op}\mathscr{S}ets$. Since $|\ \ |$ is a left Quillen functor from $\Delta^{op}\mathscr{S}ets$ to $\mathscr{K}$, all simplicial sets are cofibrant and Kelley spaces are fibrant, $|f|$ is a weak equivalence between fibrant–cofibrant objects in $\mathscr{K}$. Then $|f|$ is a left homotopy equivalence in the simplicial closed symmetric monoidal model category $\mathscr{K}$. Applying Lemma 1, we obtain that $\mathrm{Sym}^n(|f|)$ is a weak equivalence in $\mathscr{K}$ for all $n \geq 0$. Since $|\ \ |$ is monoidal and left adjoint, we have that $\mathrm{Sym}^n(|f|)$ is the same morphism as $|\mathrm{Sym}^n(f)|$. $\square$

Proposition 60. *All (trivial) cofibrations in $\Delta^{op}\mathscr{S}ets$, and all (trivial) cofibrations in $\Delta^{op}\mathscr{S}ets_*$ are symmetrizable.*

Proof. By Lemma 12, it is enough to prove the proposition in the unpointed case only. For the set of all cofibrations, since the monoidal product and colimits in $\Delta^{op}\mathscr{S}ets$ are level-wise, it is enough to prove a similar proposition in the category of sets, where cofibrations are injections. This is an easy exercise. For the set of all trivial cofibrations, we apply Lemma 59 together with Corollary 23. $\square$

Proposition 61. *All (trivial) cofibrations in $\mathscr{K}$, and all (trivial) cofibrations in $\mathscr{K}_*$ are symmetrizable.*

Proof. Since $|I_s| = I$, $|J_s| = J$, and $|\ \ |$ is a symmetric monoidal functor commuting with colimits, we see that by Proposition 60, I and J are symmetrizable. Thus we conclude by Corollary 9. $\square$

Since the sets of cofibrations and trivial cofibrations in $\Delta^{op}\mathscr{S}ets$, $\Delta^{op}\mathscr{S}ets_*$, $\mathscr{K}$, and $\mathscr{K}_*$ are symmetrizable, we can apply Theorem 25 getting λ-structures of left derived symmetric powers in the corresponding unstable homotopy categories. In the stable setting, when $\mathscr{S} = \mathrm{Spt}^{\Sigma}(\mathscr{C}, T)$, the category $\mathscr{C}$ is the category $\Delta^{op}\mathscr{S}ets_*$ of pointed simplicial sets and T is the simplicial circle S^1, i.e. the coequalizer of the two boundary morphisms $\Delta[0] \rightrightarrows \Delta[1]$, then Theorem 55 and Corollary 56 specialize to the results [3], III, 5.1, and [15], 15.5. Corollary 57 yields the λ-structure of left derived symmetric powers in the topological stable homotopy category.

11. Symmetrizable cofibrations in $\mathbb{A}^1$-homotopy theory of schemes

Now we are going to apply the main results of the paper to the Morel–Voevodsky homotopy theory of schemes over a base and prove the existence of λ-structures of left derived symmetric powers in both unstable and stable settings of that theory.

Let B be a Noetherian separated scheme of finite Krull dimension, $\mathscr{S}m/B$ the category of smooth schemes of finite type over B, and let $Pre(\mathscr{S}m/B)$ be the category of presheaves of sets on $\mathscr{S}m/B$, i.e. contravariant functors from $\mathscr{S}m/B$ to $\mathscr{S}ets$. Let $\mathscr{C}$ be the category $\Delta^{op}Pre(\mathscr{S}m/B)$ of simplicial presheaves over B. Sometimes it is convenient to think of $\mathscr{C}$ as the category $Pre(\mathscr{S}m/B \times \Delta)$ of presheaves of sets on the Cartesian product of two categories $\mathscr{S}m/B$ and Δ. If X is a smooth scheme over the base B, let $\Delta_X[n]$ be a presheaf on $\mathscr{S}m/B \times \Delta$ sending any pair $(U, [m])$ to the Cartesian product of sets $\mathrm{Hom}_{\mathscr{S}m/B}(U, X) \times \mathrm{Hom}_{\Delta}([m], [n])$. Then we get a fully faithful embedding $\mathscr{S}m/B \to \mathscr{C}$ of Yoneda type, sending X to the presheaf $\Delta_X[0]$ represented by X, and similarly on morphisms. If K is a simplicial set, i.e. a presheaf of sets on the simplicial category Δ, then it induces another presheaf on $\mathscr{S}m/B \times \Delta$ by ignoring schemes and sending a pair (U, m) to the value K_m of the functor K on the object $[m]$ in Δ. This gives a functor $\Delta^{op}\mathscr{S}ets \to \mathscr{C}$, which provides a simplicial structure on the category $\mathscr{C}$. The symmetric monoidal structure in $\mathscr{C}$ is defined section-wise, i.e. for any two simplicial presheaves X and Y the value of their product on $(U, [m])$ is the Cartesian product of the values of X and Y on $(U, [m])$.

Following Jardine, [13], we say that a morphism $f : X \to Y$ in $\mathscr{C}$ is a weak equivalence if f induces weak equivalences on stalks of the presheaves X and Y, where stalks are taken in the sense of Nisnevich (or étale) topology on the category $\mathscr{S}m/B$. Let W be the class of all weak equivalences in $\mathscr{C}$. Notice that, in spite of that $\mathscr{C}$ is a category of simplicial presheaves, the topology is needed to define weak equivalences in $\mathscr{C}$ in terms of stalks. Let also I be the set of monomorphisms of type $X \hookrightarrow \Delta_U[n]$ for some simplicial presheaf X, smooth B-scheme U and $n \geq 0$. Fix a cardinal $\beta > 2^{\alpha}$, where α is the cardinality of the morphisms in $\mathscr{S}m/B$. Let J be the set of monomorphisms $X \to Y$, which are weak equivalences and such that the cardinal of the set of n-simplices in Y is less than β for all n. One can show that the class I-cell consists of all section-wise monomorphisms of simplicial presheaves. Then $\mathscr{C}$ together with the above defined weak equivalences and monomorphisms taken as cofibrations is a simplicial left proper and cellular closed symmetric monoidal model category cofibrantly generated by the set of generating cofibrations I and the set of generating trivial cofibrations J. Actually, this is a consequence of a more general result on model structures for simplicial presheaves on a site due to Jardine, see [13]. Such constructed model structure $\mathscr{M} = (I, J, W)$ is called the injective model structure in $\mathscr{C}$.

As well as in Example 41, denote by $\mathbb{A}^1$ the simplicial motivic space represented by the affine line $\mathbb{A}^1_B$ over the base scheme B. Then $\mathbb{A}^1 \to \mathbb{1}$ is a diagonalizable interval, with the multiplication coming from the multiplication in the fibres of the structural morphism from $\mathbb{A}^1_B$ to B. The above injective model structure and the set of morphisms $S = \{X \wedge \mathbb{A}^1 \xrightarrow{\mathrm{id} \wedge \pi} X \mid X \in \mathrm{dom}(I) \cup \mathrm{codom}(I)\}$ satisfy the assumptions of the localization theorem in [8]. The corresponding left localized model structure $\mathscr{M}_{\mathbb{A}^1} = (I, J_{\mathbb{A}^1}, W_{\mathbb{A}^1})$ is one of the motivic model structures on $\mathscr{C}$, and the corresponding localization $\mathscr{C}_{\mathbb{A}^1}$ is again a simplicial left proper cellular closed symmetric monoidal model category cofibrantly generated by the same set of generating cofibrations I and the new localized set of generating trivial cofibrations $J_{\mathbb{A}^1}$. The category $\mathscr{C}_{\mathbb{A}^1}$ is called the unstable motivic model category of schemes over the base B. Its homotopy category $Ho(\mathscr{C}_{\mathbb{A}^1})$ is nothing but the unstable motivic homotopy category of schemes over B, which we denote by $\mathbf{H}(B)$.

The following result is the precise statement of Theorem A mentioned in Introduction.

Theorem 62. *Let B be a Noetherian scheme of finite Krull dimension, and let $\mathscr{C}_{\mathbb{A}^1}$ be the unstable motivic model category of schemes over B. Then all symmetric powers Sym^n preserve weak equivalences in $\mathscr{C}_{\mathbb{A}^1}$, and the corresponding left derived functors $L\mathrm{Sym}^n$ yield a λ-structure in $\mathbf{H}(B)$.*

Proof. Since cofibrations in $\mathscr{C}$ are coming section-wise from cofibrations simplicial sets, all objects are cofibrant in $\mathscr{C}$. By the same reason, and by Proposition 60, we also have that all cofibrations in $\mathscr{C}$ are symmetrizable. The class of trivial cofibrations is symmetrizable too. Indeed, let $f : X \to Y$ be a trivial cofibration $\mathscr{C}$. Since stalks of presheaves are colimits commuting with symmetric powers, the morphism $(\mathrm{Sym}^n(f))_P$ on stalks at a point P is nothing but the n-th symmetric power $\mathrm{Sym}^n(f_P)$ of the morphism f_P induced by f at P. So $(\mathrm{Sym}^n(f))_P$ is a weak equivalence of simplicial sets by Proposition 60. Since, moreover, $\mathbb{A}^1 \to \mathbb{1}$ is a diagonalizable interval and all objects are cofibrant in $\mathscr{C}$, we conclude by Theorem 42 and Theorem 25. $\square$

Remark 63. Theorem 62 holds true also in the pointed setting by Lemma 12.

Let now T be the motivic $(1,1)$-sphere. Recall that T is the $\wedge$-product of the simplicial circle, i.e. the coequalizer of the two morphisms from $\Delta[0]$ to $\Delta[1]$, and the algebraic group $\mathbb{G}_{\mathrm{m}}$ over B in the pointed category $\mathscr{C}_*$. The corresponding category of symmetric spectra $\mathscr{S} = \mathrm{Spt}^{\Sigma}((\mathscr{C}_{\mathbb{A}^1})_*, T)$, together with the corresponding stable model structure, is the category of motivic symmetric spectra over the base scheme B, and the homotopy category of $\mathscr{S}$, with regard to the stable model structure, is nothing but the Morel–Voevodsky motivic stable homotopy category over B, see [22] and [14]. We will denote it by $\mathbf{SH}(B)$.

The category $\mathscr{S} = \mathrm{Spt}^{\Sigma}((\mathscr{C}_{\mathbb{A}^1})_*, T)$ of motivic symmetric spectra has a structure of a simplicial closed symmetric monoidal model category by Hovey's result, [11]. Moreover, the simplicial suspension Σ_{S^1} induces an autoequivalence in its homotopy category $\mathbf{SH}(B)$, so that it is a triangulated category (use Section 6.5 in [10]). Then we see that the results in Proposition 53, Theorem 55, Corollary 56 and Corollary 57 hold true for symmetric spectra of simplicial sets and for motivic symmetric spectra uniformly. In other words, we have the following result (Theorem B in Introduction).

Theorem 64. *Let B be a Noetherian scheme of finite Krull dimension, and let $T = S^1 \wedge \mathbb{G}_{\mathrm{m}}$ be the motivic sphere. Symmetric powers preserve stable weak equivalences between positively cofibrant objects in the category $\mathrm{Spt}^{\Sigma}((\mathscr{C}_{\mathbb{A}^1})_*, T)$ of motivic symmetric spectra over the base B. The corresponding left derived symmetric powers $L\mathrm{Sym}^n$ exist, they are canonically isomorphic to homotopy symmetric powers and give rise to a λ-structure in $\mathbf{SH}(B)$.*

The category $\mathbf{SH}(B)$, being triangulated, can be $\mathbb{Q}$-localized getting the $\mathbb{Q}$-linear triangulated symmetric monoidal category $\mathbf{SH}(B)_{\mathbb{Q}}$. Hirschhorn's localization allows to make symmetric spectra into a $\mathbb{Q}$-linear stable model category, see Definition 3.2.14 in [1]. One can show that the λ-structure from Theorem 64 induces the λ-structure of symmetric powers with $\mathbb{Q}$-coefficients defined via idempotents in endomorphism rings, see 3.3.21 in [1]. The latest λ-structure coincides with the system of towers constructed in [5]. If now $\mathbf{SH}(B)^{\mathrm{c}}_{\mathbb{Q}}$ is the full subcategory of compact objects in $\mathbf{SH}(B)_{\mathbb{Q}}$, the λ-structure of $\mathbb{Q}$-local left derived symmetric powers induces the λ-structure in the K-theory of the triangulated category $\mathbf{SH}(B)^{\mathrm{c}}_{\mathbb{Q}}$ considered in [6].

Acknowledgments

The authors are grateful to Peter May for helpful suggestions and stimulating interest to this work. We also thank Bruno Kahn, Dmitry Kaledin, Roman Mikhailov, Oliver Röndigs, Alexander Smirnov, Vladimir Voevodsky and Chuck Weibel for useful discussions along the theme of this paper. We acknowledge the effort of the anonymous referee which helped to improve the text of the paper. The research is done in the framework of the EPSRC grant EP/I034017/1 "Lambda-structures in stable categories". The first named author acknowledges the support of the grants MK-5215.2015.1, RFBR 13-01-12420, 14-01-00178 and the subsidy granted to the HSE by the Government of the Russian Federation for the implementation of the Global Competitiveness Program. The second named author is grateful to the Institute for Advanced Study in Princeton for the support and hospitality in 2004–2006, where and when this work began.

Appendix A. Categorical v.s. geometrical symmetric powers

Let k be a field, and let $\mathscr{S}ch$ be the category of separated schemes of finite type over k. Let $\mathscr{S}m$ be the full subcategory of smooth schemes in $\mathscr{S}ch$, and let $\mathscr{C}$ be the category of simplicial presheaves on $\mathscr{S}m$. The fully faithful embedding of $\mathscr{S}m$ into $\mathscr{C}$ can be extended to $\mathscr{S}ch$, sending a scheme X from $\mathscr{S}ch$ to the functor $h_X = \Delta_X[0]$, and similarly on morphisms. Let $E(X)$ be the motivic symmetric spectrum of the motivic space h_X. If any finite subset in X is contained in an affine open subscheme in X, the n-th symmetric power $\mathrm{Sym}^n(X)$ exists as an object in $\mathscr{S}ch$, and the rational homotopy type of the motivic spectrum $\mathrm{Sym}^n(E(X))$ is the same as the rational homotopy type of the motivic spectrum $E(\mathrm{Sym}^n(X))$, see [17].

In the unstable motivic category the situation is more complicated, as the rational unstable motivic homotopy theory is not yet in place. Working integrally, the homotopy type of the categorical n-th symmetric power $\mathrm{Sym}^n(h_X)$ of the motivic space h_X is not the same as the homotopy type of the motivic space $h_{\mathrm{Sym}^n(X)}$. The comparison of these two objects is a question of critical importance, since its understanding would provide the geometrical meaning to our categorical approach to symmetric powers in the

$\mathbb{A}^1$-homotopy setup. Below we consider a certain argument, which gives a flavour what the discrepancy between two homotopy types in question might depend on.

First we should look at the category of sets $\mathscr{S}ets$ with the discrete topology on it. Presheaves on $\mathscr{S}ets$ have one stalk only. Therefore, if X is a set and G is a finite group acting on X, it is easy to show that the canonical map from h_X/G to $h_{X/G}$ is an isomorphism. So, all is fine in the simplest possible case.

Let now X be a scheme from $\mathscr{S}ch$ and let G be a finite group acting on X. Suppose X can be covered by G-invariant affine open subschemes, so that the quotient X/G exists in $\mathscr{S}ch$. The group G acts freely on X if the canonical morphism

$$\pi : X \to X/G$$

is étale. Let also

$$\alpha : h_X/G \to h_{X/G}$$

be the obvious canonical morphism in $\mathscr{C}$. In case of symmetric powers, X must be the n-th power of a scheme, and G must be the symmetric group Σ_n permuting factors in X.

Proposition 65. *If G acts freely on X, the canonical morphism α is a weak equivalence in the étale injective model structure on $\mathscr{C}$.*

Proof. To prove the proposition it is enough to show that α induces isomorphisms on spectra of strictly Henselian rings. Let R be a strictly Henselian local ring, $\mathfrak{m}$ be the maximal ideal in it and $l = R/\mathfrak{m}$ be the corresponding residue field. All we need to show is that the canonical morphism of sets

$$\alpha_R : X(R)/G \to (X/G)(R)$$

is an isomorphism. Let $\mathscr{A}_R$ be the category of étale algebras over R and let $\mathscr{A}_l$ be the category of étale algebras over l. As R is Henselian, the residue homomorphism $R \to l$ induces an equivalence of categories $\Psi : \mathscr{A}_R \to \mathscr{A}_l$. Let

$$f : \mathrm{Spec}(R) \to X/G$$

be an element in $(X/G)(R)$. The preimage of f under the morphism π is a set of R-points of the étale R-algebra S, where $\mathrm{Spec}(S) \to X$ is the pull-back of f with respect to the morphism π. Let

$$\bar{f} : \mathrm{Spec}(l) \to X/G$$

be the precomposition of f with the morphism $\mathrm{Spec}(l) \to \mathrm{Spec}(R)$, and let

$$\mathrm{Spec}(L) \to X$$

be the pull-back of $\bar{f}$ with regard to π. As Ψ is an equivalence of categories,

$$\alpha_R^{-1}(f) = \alpha_l^{-1}(\bar{f}) ,$$

where α_l is the morphism from $X(l)/G$ to $(X/G)(l)$. In other words, $\alpha_R^{-1}(f)$ is in bijection to l-points of the étale l-algebra L. Since R is strictly Henselian, the residue field l is separably closed. This gives that L is isomorphic to the product of n copies of l, where n is the order of G, and G acts freely on $\mathrm{Spec}(L)$. Then the quotient of the set of l-points of X by G can be identified with l-points of X/G. Therefore, the quotient of the set of R-points of X by G can be identified with R-points of X/G. Hence, α_R is a bijection. □

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